Stat 8931, Fall 2005
Homework 5

Homework Problem 5 Question Show that the Gibbs sampler for Homework problem 2 is geometrically ergodic.

The observed data are IID Normal(μ, λ^{-1}) where the prior distributions for the mean μ and precision λ have them independently distributed

$$\begin{aligned} \mu &\sim \text{Normal}(m, l^{-1}) \\ \lambda &\sim \text{Gamma}(a, b) \end{aligned}$$

here m, l, a and b are hyperparameters that are fixed known numbers (which are adjusted to represent the subjective opinion of the person whose prior it is). Problem 2 gave actual values for the sample size n and the hyperparameters. Here we would like our proof to work for any $n > 1$, any real m, and any positive real l, a, and b.

The Gibbs sampler samples from the "full conditionals"

$$\begin{aligned} \mu \mid \lambda &\sim \text{Normal}\left(\frac{lm + n\lambda\hat{\mu}_n}{l + n\lambda}, \frac{1}{l + n\lambda}\right) \\ \lambda \mid \mu &\sim \text{Gamma}\left(a + \frac{n}{2}, b + \frac{n\hat{\sigma}_n^2 + n(\hat{\mu}_n - \mu)^2}{2}\right) \end{aligned}$$

where $\hat{\mu}_n$ and $\hat{\sigma}_n^2$ are the MLE for μ and $\sigma^2 = 1/\lambda$. (We are using the parameterization of the gamma distribution where b is the rate parameter, not the scale parameter).

The proof that this is a T-chain follows the notes closely; you do not need to reproduce that. Hence you may assume that every compact set of the state space, which is $(-\infty, \infty) \times (0, \infty)$ is petite. Note that this means, by the Heine-Borel theorem that every *closed* and bounded subset of the state space is petite (the "closed is important).

This the actual assignment comes down to verifying a geometric drift condition (Sections 4.7.5 and 4.7.6 in the 1998 notes on the web page, Section 15.2.2 and 15.2.3 in Meyn and Tweedie).

Verify the sampler for a fixed scan (composition). You may verify geometric ergodicity for whichever scan (composition) order is more mathematically convenient.

Hint: Your drift function $V(\mu, \lambda)$ must be unbounded off petite sets. You must take its conditional expectation first with respect to the second update in the scan order, then with respect to the first. What sort of functions have nice closed form integrals with respect to these distributions?