Stat 8501 Lecture Notes

Convergence in Probability and Almost Surely

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Let $X_1, X_2, \ldots$ be a sequence of random variables and X another random variable, all defined on the same probability space. We say X_n *converges in probability to* X, written

$$X_n \xrightarrow{P} X \tag{1}$$

if for every $\varepsilon > 0$

$$\lim_{n \to \infty} P(|X_n - X| \ge \varepsilon) = 0.$$

We say X_n *converges almost surely to* X, written

$$X_n \xrightarrow{\text{a.s.}} X \tag{2}$$

if there exists a set A having probability one such that

$$\lim_{n \to \infty} X_n(\omega) \to X(\omega), \qquad \omega \in A. \tag{3}$$

Define

$$B_{\varepsilon n} = \{\, \omega \in \Omega : |X_n(\omega) - X(\omega)| \le \varepsilon \,\},$$

so (1) is equivalent to $P(B_{\varepsilon n}) \to 1$ as $n \to \infty$ holding for every $\varepsilon > 0$.

Also define

$$C_{\varepsilon m n} = \bigcap_{i=m}^{n} B_{\varepsilon i}.$$

where n is allowed to be any integer greater than m or is allowed to be ∞. Then $C_{\varepsilon m n} \downarrow C_{\varepsilon m \infty}$ as $n \to \infty$. So by continuity of probability

$$\lim_{n \to \infty} P(C_{\varepsilon m n}) = P(C_{\varepsilon m \infty}). \tag{4}$$

Then define

$$C_{\varepsilon \infty \infty} = \cap_{m=1}^{\infty} C_{\varepsilon m \infty}.$$

Then $C_{\varepsilon m \infty} \downarrow C_{\varepsilon \infty \infty}$ as $n \to \infty$. So by continuity of probability

$$\lim_{m \to \infty} P(C_{\varepsilon m \infty}) = P(C_{\varepsilon \infty \infty}). \tag{5}$$

The limit in (3) holds for some ω if and only if for every $\varepsilon > 0$ there exists an m such that $\omega \in C_{\varepsilon k \infty}$ for all $k \ge m$. Hence the limit in (3) holds for some ω if and only if for every $\varepsilon > 0$ we have $\omega \in C_{\varepsilon\infty\infty}$. So (2) is equivalent to $P(C_{\varepsilon\infty\infty}) = 1$ holding for every $\varepsilon > 0$.

By (4) and (5) we have $P(C_{\varepsilon\infty\infty}) = 1$ if and only if

$$\lim_{m\to\infty} \left[\lim_{n\to\infty} P(C_{\varepsilon mn}) \right] = 1.$$

Since $P(C_{\varepsilon mn})$ can be calculated using only finite-dimensional distributions, this gives a characterization of almost sure convergence that does not involve infinite-dimensional sample paths.