Stat 5101 Final Exam **May 9, 2022**

Name ______________________________ Student ID ____________________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. You may use a calculator. No other electronic devices are allowed.

Put all of your work on this test form (use the backs of pages if necessary). Show your work or give an explanation of your answer. No credit for numbers or formulas with no indication of where they came from. Leave no undone sums or integrals for probabilities or expectations in your answers. But other than that requirement, there is no unique "correct" simplification for any answer. Any correct (and explained) answer gets full credit unless the question explicitly states otherwise.

Abbreviations used:

IID for independent and identically distributed

PDF for probability density function

$\text{Gam}(\alpha, \lambda)$ for the gamma distribution with parameters α and λ

$\text{NegBin}(r, p)$ for the negative binomial distribution with parameters r and p

$\text{Beta}(\alpha_1, \alpha_2)$ for the beta distribution with parameters α_1 and α_2

The points for the questions total to 200. There are 9 pages and 8 problems.

1. [25 pts.] Suppose Suppose X_1, ..., X_N are IID $\text{Gam}(\alpha, \lambda)$ random variables, where N is a $\text{NegBin}(r, p)$ random variable that is independent of all of the X_i. Let

$$Y = \sum_{i=1}^{N} X_i,$$

with the convention that $N = 0$ means $Y = 0$.

(a) Find $E(Y)$.

(b) Find $\text{var}(Y)$.

2. [25 pts.] Define

$$h_\theta(x) = \frac{1}{1 + \cos(x) + \theta x^2 + x^4}, \qquad 0 < x < \infty.$$

(a) For what values of the positive real parameter θ does there exist a constant $c(\theta)$ that

$$f_\theta(x) = c(\theta) h_\theta(x), \qquad 0 < x < \infty,$$

is a PDF?

(b) If X is a random variable having this PDF, for what values of $\theta > 0$ and $\beta > 0$ does the expectation of X^β exist?

3. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID Beta(θ, θ) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{$*$}$$

What is the approximate normal distribution of $\log(\overline{X}_n)$ when n is large?

4. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID $\text{Gam}(\theta, \theta^3)$ random variables, where $\theta > 0$ is a real parameter. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large n with variance that is a constant function of the parameter? As usual,

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

5. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f_\theta(x, y) = \frac{(\theta x + y^2)^2 e^{-x-y}}{24 + 4\theta + 2\theta^2}, \qquad 0 < x < \infty,\ 0 < y < \infty,$$

where $\theta > 0$ is a real parameter.

Hint: the theorem associated with the gamma distribution (brand name distributions handout).

(a) Find the conditional PDF of X given Y.

(b) Find the conditional expectation of X given Y.

6. [25 pts.] Suppose X_1, X_2, ... is a sequence of random variables having

$$\begin{aligned} E(X_i) &= \mu, && \text{for all } i \\ \operatorname{cov}(X_i, X_j) &= \rho^{|i-j|}\sigma^2, && \text{for all } i \text{ and } j \end{aligned}$$

(so this sequence is a weakly stationary time series).

(a) What is $E(X_1 - X_i)$ for arbitrary i?

(b) What is $\operatorname{cov}(X_1 - X_i, X_1 - X_j)$ for arbitrary i and j?

7. [25 pts.] The following defines the PDF of a probability distribution

$$f(x) = \frac{\sin(x)}{2}, \qquad 0 < x < \pi.$$

Hint: You can express your answers using inverse trignometric functions (if necessary), arcsine, arccosine, arctangent, and so forth.

(a) Find the DF of this distribution. Be sure to define the DF on the whole real line.

(b) Find the quantile function of this distribution. Be sure to specify its domain as well as the formula.

8. [25 pts.] Suppose X is a Beta(α_1, α_2) random variable. Find the distribution of $Y = X^\beta$ for any real β. Any function definition includes a domain as well as a formula.

   Hint: there is something odd about the case $\beta = 0$.