Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

11-1. Suppose the random vector \((X, Y)\) has variance matrix
\[
\begin{pmatrix}
4 & -3 \\
-3 & 9
\end{pmatrix}
\]
Find \(\text{sd}(X)\), \(\text{sd}(Y)\), and \(\text{cor}(X, Y)\). Note: \(\text{cor}(X, Y)\) is correlation not covariance.

11-2. Suppose \(\text{sd}(X) = 5\), \(\text{sd}(Y) = 7\), and \(\text{cor}(X, Y) = 2/3\). Find the variance matrix of the random vector \((X, Y)\).

11-3. Suppose \(X\) has the Poisson distribution with mean 100.
   
   (a) Calculate \(\Pr(X < 80)\) exactly.
   
   (b) Calculate \(\Pr(X < 80)\) using the normal approximation without correction for continuity.
   
   (c) Calculate \(\Pr(X < 80)\) using the normal approximation with correction for continuity.
   
   (d) Which of (b) and (c) is closer to correct?
   
   (e) Calculate \(\Pr(X > 120)\) exactly.
   
   (f) Calculate \(\Pr(X > 120)\) using the normal approximation without correction for continuity.
   
   (g) Calculate \(\Pr(X > 120)\) using the normal approximation with correction for continuity.
   
   (h) Which of (f) and (g) is closer to correct?

Be careful about weak and strict inequality.

11-4. Suppose \(X_1, \ldots, X_{40}\) are IID random variables having the exponential distribution with rate parameter one. Let \(Y = X_1 + \cdots + X_{40}\).
   
   (a) Calculate \(\Pr(Y < 25)\) exactly.
(b) Calculate $\Pr(Y < 25)$ using the normal approximation.

(c) Calculate $\Pr(Y > 55)$ exactly.

(d) Calculate $\Pr(Y > 55)$ using the normal approximation.

11-5. Suppose $X_1, \ldots, X_{50}$ are IID random variables having mean 10 and standard deviation 5. Let $\overline{X}_n = (X_1 + \cdots + X_n)/n$.

(a) Calculate $\Pr(\overline{X}_n < 9)$ using the normal approximation.

(b) Calculate $\Pr(\overline{X}_n > 11)$ using the normal approximation.

11-6. Suppose $X_1, \ldots, X_n$ are IID random variables having mean $\mu$ and standard deviation $\sigma > 0$. Let $\overline{X}_n = (X_1 + \cdots + X_n)/n$. Find a number $c$, which will be a function of $\sigma$ and $n$, such that

$$\Pr(|\overline{X}_n - \mu| > c) \approx 0.05,$$

where the $\approx$ means approximately equal using the normal approximation.

11-7. Suppose $X_1, X_2, \ldots$ is a sequence of IID random variables having mean $\mu$ and standard deviation $\sigma > 0$. Let $\overline{X}_n = (X_1 + \cdots + X_n)/n$. Does

$$\frac{n(\overline{X}_n - \mu)^2}{\sigma^2}$$

converge in distribution? If so, to what distribution does it converge?

Hint: CLT and continuous mapping theorem.

11-8. Suppose $X_1, X_2, \ldots$, is a sequence of random variables, $\theta$ is a constant, and

$$\sqrt{n}(X_n - \theta) \xrightarrow{D} Y$$

where $Y$ is any random variable. Show that this implies

$$X_n \xrightarrow{P} \theta.$$

Hint: Slutsky’s theorem.

11-9. Suppose $X_1, X_2, \ldots$, is a sequence of IID random variables, having mean $\mu$ and standard deviation $\sigma > 0$. Suppose

$$S_n = g_n(X_1, \ldots, X_n)$$

is some function of the data such that

$$S_n \xrightarrow{P} \sigma.$$

Show that

$$\frac{\sqrt{n}(\overline{X}_n - \mu)}{S_n} \xrightarrow{D} N(0, 1).$$

Hint: CLT and Slutsky’s theorem.