1. [20 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. discrete random variables with density
\[
f_p(x) = \binom{x + m - 1}{m - 1} p^m (1 - p)^x, \quad x = 0, 1, 2, \ldots
\]
where $m$ is a known nonnegative integer constant and $p$ is an unknown parameter satisfying $0 < p < 1$.

(a) Find the MLE of $p$.
(b) Find the observed or expected Fisher information $p$ (either will do).
(c) Find an asymptotic 95% confidence interval for $p$.

2. [20 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. from the distribution with density
\[
f(x) = \theta x^{-\theta - 1}, \quad x > 1,
\]
where $\theta > 0$ is an unknown parameter. Suppose our prior distribution for the parameter $\theta$ is $\text{Exp}(\lambda)$, where $\lambda$ is a known number (hyperparameter of the prior).

(a) Find the posterior density of $\theta$.
(b) Find the posterior mean of $\theta$.

3. [20 pts.] Suppose $X_1, X_2, \ldots$ are i. i. d. from a statistical model having a single parameter $\theta > 0$. I do not tell you anything about the model other
than that the MLE $\hat{\theta}_n$ exists and satisfies the conditions for the usual asymptotics to hold and that the Fisher information is

$$I_1(\theta) = \frac{\theta^2}{2}$$

(a) Find the Fisher information for the parameter $\varphi = \log(\theta)$.

(b) Find the asymptotic distribution of the MLE $\hat{\varphi}_n$ of the parameter $\varphi$.

4. [20 pts.] Suppose $X_1, X_2, \ldots$ are i. i. d. $\mathcal{N}(\mu, \sigma^2)$ distribution, where $\mu$ and $\sigma^2$ are unknown parameters, and suppose $S^2_n$ is the usual sample variance

$$S^2_n = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$ 

Perform an exact test of the hypotheses

$$H_0 : \sigma^2 \leq 1$$
$$H_A : \sigma^2 > 1$$

corresponding to sample size $n = 10$ and sample variance $S^2_n = 2.3$. Give the $P$-value for the test and also say whether $H_0$ is accepted or rejected at the .05 level of significance.

5. [20 pts.] Suppose $X_1, X_2, \ldots$ are i. i. d. Geo($p$) random variables and, as usual, $\bar{X}_n$ denotes the sample mean. Perform an asymptotic (large sample) test of the hypotheses

$$H_0 : p = \frac{1}{4}$$
$$H_A : p \neq \frac{1}{4}$$

corresponding to sample size $n = 100$ and sample mean $\bar{X}_n = 3.6$. Give the $P$-value for the test and also say whether $H_0$ is accepted or rejected at the .05 level of significance.