1. [25 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. Gam($\alpha, \lambda$) random variables and, as usual, $\bar{X}_n$ denotes the sample mean. What is the asymptotic distribution of $\log(\bar{X}_n)$? You must give the parameters of the asymptotic distribution as functions of $\alpha$ and $\lambda$ for full credit.

2. [25 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. Beta($\theta, 1 - \theta$) random variables, where $0 < \theta < 1$. Find a method of moments estimator of $\theta$ and its asymptotic distribution. You must give the parameters of the asymptotic distribution as functions of $\theta$ for full credit.

3. [25 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. Beta($s, t$) random variables. Perform an asymptotic (large sample) test of the hypotheses

$$H_0 : s = t$$
$$H_A : s \neq t$$

corresponding to sample size $n = 100$, sample mean $\bar{X}_n = 0.57$, and sample variance $S^2_n = 0.036$. Give the $P$-value for the test and also say whether $H_0$ is accepted or rejected at the .05 level of significance.

4. [25 pts.] Suppose $X_1, X_2, \ldots, X_n$ are i. i. d. $\mathcal{N}(\mu, 25)$ random variables, and we observe $\bar{X}_n = 31.2$ for sample size $n = 16$. We want to do a Bayesian analysis with a $\mathcal{N}(20, 10)$ prior distribution for $\mu$. Find a 95% HPD region for $\mu$. 
5. [25 pts.] In Problem 6-6 in the notes, the part of the posted solution was

\[
x_{\text{low}} \leftarrow \text{ifelse}(x < 11, x - 11, 0) \\
x_{\text{hig}} \leftarrow \text{ifelse}(x < 11, 0, x - 11) \\
\text{out} \leftarrow \text{lm}(y \sim x_{\text{low}} + x_{\text{hig}}) \\
\text{summary(out)}
\]

Recall that this fits a regression model with regression function

\[
h(x) = \begin{cases} 
\alpha + \beta_1(x - 11), & x \leq 11 \\
\alpha + \beta_2(x - 11), & x \geq 11
\end{cases}
\]

Explain what two models are involved in the following printout.

\[
> \text{out.too} \leftarrow \text{lm}(y \sim x) \\
> \text{anova(out.too, out)}
\]

Analysis of Variance Table

<table>
<thead>
<tr>
<th>Model 1: y ~ x</th>
<th>Model 2: y ~ x_{\text{low}} + x_{\text{hig}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Res.Df</td>
<td>19</td>
</tr>
<tr>
<td>Res.Sum Sq</td>
<td>195.966</td>
</tr>
<tr>
<td>Df</td>
<td>1</td>
</tr>
<tr>
<td>Sum Sq</td>
<td>72.961</td>
</tr>
<tr>
<td>F value</td>
<td>10.677</td>
</tr>
<tr>
<td>Pr(&gt;F)</td>
<td>0.004277 **</td>
</tr>
</tbody>
</table>

Also explain why these are nested models and what conclusion about the fit of these two models can be drawn from the printout.

6. [25 pts.] Suppose \(X_1, X_2, \ldots, X_n\) are i.i.d. Beta(\(\theta, 1\)) random variables and the prior distribution of \(\theta\) is Gam(\(\alpha, \lambda\)). Find the posterior distribution of \(\theta\).

7. [25 pts.] Suppose we have regression data with variables \(x\) and \(y\) and fit a quadratic model

\[
\text{out} \leftarrow \text{lm}(y \sim x + I(x^2)) \\
\text{options(show.signif.stars=FALSE)} \\
\text{summary(out)}
\]

getting the following (partial) output
Coefficients:

|                | Estimate | Std. Error | t value | Pr(>|t|) |
|----------------|----------|------------|---------|----------|
| (Intercept)    | -0.544281| 0.767327   | -0.709  | 0.488    |
| x              | 1.216011 | 0.168287   | 7.226   | 1.42e-06 |
| I(x^2)         | -0.011464| 0.007784   | -1.473  | 0.159    |

Residual standard error: 1.031 on 17 degrees of freedom
Multiple R-Squared: 0.9723, Adjusted R-squared: 0.969

Give a 90% confidence interval for the coefficient of \( x^2 \) in the regression function.

8. [25 pts.] Suppose \( X_1, X_2, \ldots, X_n \) are i. i. d. random variables with density

\[
f(x) = \frac{2}{\pi} \sqrt{1 - x^2}, \quad -1 < x < +1
\]

shown below

What is the asymptotic distribution of the sample median?