Problem L5-3

Let \( Y_i = X^k_i \), then \( Y_1, Y_2, \ldots \) is a sequence of independent identically distributed random variables (functions of independent random variables are independent by Theorem 13 of Chapter 3 in Lindgren) with expectation

\[ \mu_Y = E(Y_i) = E(X^k). \]

Then the LLN says

\[ \overline{Y}_n \xrightarrow{P} \mu_Y \]

but this is just other notation for

\[ \frac{1}{n} \sum_{i=1}^{n} X^k_i \xrightarrow{P} E(X^k). \]

Problem L5-6

Write \( Y \) for the weight of the 100 booklets. Then

\[ E(Y) = 100 \]
\[ \text{var}(Y) = 100 \times .02^2 = .04 \]

so

\[ P(Y > 100.5) = 1 - P(Y < 100.5) = 1 - \Phi \left( \frac{100.5 - 100}{\sqrt{100 \times .02}} \right) = 1 - \Phi(2.5) = .0062 \]

Problem L5-9

Let \( Y \sim \mathcal{U}(-0.5, 0.5) \) be one error, then from the appendix on brand name distributions

\[ E(Y) = 0 \]
\[ \text{var}(Y) = \frac{1}{12} \]

If \( W \) is the sum of \( n \) i. i. d. such errors then

\[ E(W) = 0 \]
\[ \text{var}(W) = \frac{n}{12} \]
Thus

\[
P \left( |W| < \sqrt{n}/2 \right) = P \left( -\sqrt{n}/2 < W < \sqrt{n}/2 \right)
\]

\[
= \Phi \left( \frac{\sqrt{n}/2 - 0}{\sqrt{n}/12} \right) - \Phi \left( -\frac{\sqrt{n}/2 - 0}{\sqrt{n}/12} \right)
\]

\[
= 1 - 2\Phi(-\sqrt{3})
\]

\[
= 0.9167355
\]

**Problem L6-13**

By direct count, the probability of a sum of 5 or less rolling a pair of dice is 5/18. Thus, if \( Y \) is the number of such rolls in 72 tries, \( Y \sim \text{Bin}(72, 5/18) \), and

\[
E(Y) = 72 \times \frac{5}{18} = 20
\]

\[
\text{var}(Y) = 72 \times \frac{5}{18} \times \frac{13}{18} = 14.4444
\]

\[
\text{sd}(Y) = \sqrt{14.4444} = 3.8006
\]

So, using a continuity correction,

\[
P(Y \geq 28) = 1 - \Phi \left( \frac{27 + 0.5 - 20}{3.8006} \right) = .0242
\]

**Problem L6-86**

From a picture of the triangular density, the two inside intervals have three times the probability of the outside intervals. Thus the probabilities of the intervals are \(1/8\), \(3/8\), \(3/8\), and \(1/8\).

Let \( X_1, X_2, X_3, \) and \( X_4 \) be the counts in the cells \((1, 2, 2, 1)\), then this is a multinomial random vector and the probability of these counts is

\[
\left( \begin{array}{c}
\frac{n}{x_1, x_2, x_3, x_4}
\end{array} \right) p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4} = \frac{6!}{1! 2! 2! 1!} \left( \frac{1}{8} \right)^1 \left( \frac{3}{8} \right)^2 \left( \frac{3}{8} \right)^2 \left( \frac{1}{8} \right)^1
\]

\[
= 180 \cdot \frac{3^4}{8^6}
\]

\[
= 0.0556183
\]

**Problem L12-12**

Since it is a linear transformation of a multivariate normal random vector, \((X, Y)\) is also multivariate normal with mean vector zero because

\[
E(X) = E(U) + 2E(V) = 0
\]

\[
E(Y) = 3E(U) - E(V) = 0
\]
and variance matrix $M$ with components

$$m_{11} = \text{var}(X)$$
$$= \text{var}(U + 2V)$$
$$= \text{var}(U) + 4\text{var}(V)$$
$$= 5$$

$$m_{22} = \text{var}(Y)$$
$$= \text{var}(3U - V)$$
$$= 9\text{var}(U) + \text{var}(V)$$
$$= 10$$

$$m_{12} = \text{cov}(X, Y)$$
$$= \text{cov}(U + 2V, 3U - V)$$
$$= 3\text{var}(U) - 2\text{var}(V)$$
$$= 1$$

$$m_{21} = m_{12}$$

**Problem N5-7**

From the variance formula for the multinomial in the appendix on brand name distributions

$$\text{var}(X_i - X_j) = \text{var}(X_i) + \text{var}(X_j) - 2\text{cov}(X_i, X_j)$$
$$= np_i(1 - p_i) + np_j(1 - p_j) + 2np_ip_j$$
$$= n[p_i + p_j - (p_i - p_j)^2]$$

**Problem N5-10**

The problem is to specialize the formula

$$f_X(x) = \frac{1}{(2\pi)^{n/2} \det(M)^{1/2}} \exp \left(-\frac{1}{2}(x - \mu)'M^{-1}(x - \mu)\right)$$

for the density of the multivariate normal to the two-dimensional case, when the mean vector is

$$\mu = \left(\begin{array}{c} \mu_X \\ \mu_Y \end{array} \right)$$

and the variance matrix is

$$M = \left(\begin{array}{cc} \sigma_X^2 & \rho \sigma_X \sigma_Y \\ \rho \sigma_X \sigma_Y & \sigma_Y^2 \end{array} \right)$$

Using the hints

$$\det(M) = \sigma_X^2 \sigma_Y^2 (1 - \rho^2)$$
and

$$M^{-1} = \frac{1}{\sigma_X^2 \sigma_Y^2 (1 - \rho^2)} \begin{pmatrix} \sigma_Y^2 & -\rho \sigma_X \sigma_Y \\ -\rho \sigma_X \sigma_Y & \sigma_X^2 \end{pmatrix}$$

$$= \frac{1}{(1 - \rho^2)} \begin{pmatrix} \frac{1}{\sigma_X^2} & -\frac{\rho}{\sigma_X \sigma_Y} \\ -\frac{\rho}{\sigma_X \sigma_Y} & \frac{1}{\sigma_Y^2} \end{pmatrix}$$

The constant part of the density is now done

$$\frac{1}{(2\pi)^{n/2} \det(M)^{1/2}} = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1 - \rho^2}}$$

because $n = 2$. So the only thing left is to match up the quadratic form in the exponent.

In general a quadratic form is written out explicitly in terms of components as

$$z' Az = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_i z_j$$

$$= \sum_{i=1}^{n} a_{ii} z_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} z_i z_j$$

In this case the quadratic form in the exponent is

$$(x - \mu)' M^{-1} (x - \mu)$$

$$= \frac{1}{(1 - \rho^2)} \left( \frac{(x - \mu)^2}{\sigma_X^2} + \frac{(y - \mu)^2}{\sigma_Y^2} - \frac{\rho(x - \mu)(y - \mu)}{\sigma_X \sigma_Y} \right)$$

which is the quadratic form in the formula to be proved. So we’re done.

**Problem N5-11**

In this case the elements of the partitioned variance matrix are all scalars

- $M_{11} = \sigma_X^2$
- $M_{12} = \rho \sigma_X \sigma_Y$
- $M_{22} = \sigma_Y^2$
- $M_{22}^{-1} = \frac{1}{\sigma_Y^2}$
Hence
\[ E(X | Y) = \mu_1 + M_{12}M_{22}^{-1}(X_2 - \mu_2) \]
\[ = \mu_X + \rho \sigma_X \sigma_Y \cdot \frac{1}{\sigma_Y^2} (Y - \mu_Y) \]
\[ = \mu_X + \rho \frac{\sigma_X}{\sigma_Y} (Y - \mu_Y) \]
\[ \text{var}(X | Y) = M_{11} - M_{12}M_{22}^{-1}M_{21} \]
\[ = \sigma_X^2 - \rho \sigma_X \sigma_Y \cdot \frac{1}{\sigma_Y^2} \rho \sigma_X \sigma_Y \]
\[ = \sigma_X^2 (1 - \rho^2) \]

**Problem N5-12**

We are to calculate \( P\{q(X) < d\} \) for given \( d \), where
\[ q(x) = (x - \mu)'M^{-1}(x - \mu) \]
and
\[ X \sim \mathcal{N}(\mu, M) \]

Now Problem 12-32 in Lindgren referred to in the hint says almost the same what we want
\[ q_2(Y) = Y'M^{-1}Y \sim \chi^2(p) \]
where
\[ Y \sim \mathcal{N}(0, M) \]

The only differences are (1) we have no means subtracted off in \( q_2 \) and (2) \( Y \) has mean zero. However,
\[ q(X) = q_2(X - \mu) \]
and
\[ X - \mu \sim \mathcal{N}(0, M) \]
so we can apply the 12-32 to this problem obtaining
\[ q(X) \sim \chi^2(p) \]

Thus
\[ P\{q(X) < d\} = F(d), \]
where \( F \) is the the c. d. f. of the \( \chi^2(p) \) distribution.

**Problem N5-13**

(a) Write
\[ Z = \begin{pmatrix} U - V \\ V - W \end{pmatrix} \]
Then
\[ Z = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} \]

thus is a linear transformation of multivariate normal, hence multivariate normal with
\[ E(Z) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

and
\[ \text{var}(Z) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

(b) From the formula for the variance,
\[ \text{var}(Z_1) = \text{var}(Z_2) = 2 \]

and
\[ \text{cor}(Z_1, Z_2) = -\frac{1}{2} \]

Thus the conditional distribution of \( Z_1 \) given \( Z_2 \) is normal with mean
\[ E(Z_1 \mid Z_2) = -\frac{1}{2} \cdot Z_2 \]

and variance
\[ \text{var}(Z_1 \mid Z_2) = 2 \left[ 1 - \left( -\frac{1}{2} \right)^2 \right] = \frac{3}{2} \]