Statistics 5101, Fall 2000, Geyer
Homework Solutions #4

Problem 4-8
(a) This density is symmetric about 0, which is thus the mean.
(b) This density is symmetric about 0, which is thus the mean.
(c) This density is symmetric about 1/2, which is thus the mean.

Problem 4-44
(a) \[
\text{var}(2X + 3Y - Z) = 2^2 \text{var}(X) + 3^2 \text{var}(Y) + 1^2 \text{var}(Z) = 14
\]
(b) \[
\text{cov}(X - 2Y, 3X + Y + 2Z) = \text{cov}(X, 3X) + \text{cov}(-2Y, Y) = 3 \text{var}(X) - 2 \text{var}(Y) = 1
\]
The other covariances vanish, because \(X\), \(Y\) and \(Z\) are independent.

Problem 4-49
\[
\text{cov}(X + Y, X - Y) = \text{cov}(X, X - Y) + \text{cov}(Y, X - Y)
\]
\[
= \text{cov}(X, X) - \text{cov}(X, Y) + \text{cov}(Y, X) - \text{cov}(Y, Y)
\]
\[
= \text{cov}(X, X) - \text{cov}(Y, Y)
\]
\[
= \text{var}(X) - \text{var}(Y)
\]
and this equals zero if and only if \(\sigma_X = \sigma_Y\).

Problem N2-3
Take \(a = 1\) and \(b = -1\) in Theorem 2.1 (linearity of expectation).
Problem N2-5

\[ E(X_n) = E\left( \frac{X_1 + \cdots + X_n}{n} \right) \]
\[ = \frac{1}{n} \left[ E(X_1) + \cdots + E(X_n) \right] \]
\[ = \frac{1}{n} \cdot n\mu \]
\[ = \mu \]

Problem N2-10

There are two things to be proved. First, since \( X - a \) and \( a - X \) are equal in distribution, they have the same moments, in particular,

\[ E(X - a) = E(a - X) \]
\[ E(X) - a = a - E(X) \]
\[ 2E(X) = 2a \]
\[ E(X) = a \]

That proves the first part.

The second part starts the same way except with \( k \)-th moments for \( k \) odd.

\[ E\{(X - a)^3\} = E\{(a - X)^3\} \]
\[ E\{(X - a)^3\} = E\{-(X - a)^3\} \]
\[ E\{(X - a)^3\} = -E\{(X - a)^3\} \]

because \((-1)^k = -1\) if \( k \) is odd. Since the only number that is its own negative is zero,

\[ E\{(X - a)^3\} = 0, \]

and this is what was to be proved because \( \mu = a \) by the first part, so this is the \( k \)-th central moment.

Problem N2-11

(a) The inverse transformation \( X = a + Y \) has derivative 1, so

\[ f_Y(y) = f_X(a + y) \]

(b) The inverse transformation \( X = a - Z \) has derivative \(-1\), so

\[ f_Z(z) = f_X(a - z) \]
(c) The two functions defined in parts (a) and (b) are the same if and only if they have the same values for the same argument, say $t$

\[
\begin{align*}
    f_Y(t) &= f_Z(t) \\
    f_X(a + t) &= f_X(a - t)
\end{align*}
\]

which is what was to be proved.

**Problem N2-12**

**(all parts)** Since these are symmetric distributions, the medians are the same as the means calculated in Problem 4-8.

**Problem N2-14**

**(a)** Since $X$ is either zero or one and $0^k = 0$ and $1^k = 1$ for all $k$, it follows that $X^k = X$ for all $k$, and

\[
E(X^k) = E(X) = \mu
\]

**(b)** Since $0 \leq X \leq 1$, it follows that

\[
0 \leq E(X) \leq 1
\]

by monotonicity of probability (Theorem 2.8 in the notes).

**(c)**

\[
\text{var}(X) = E(X^2) - E(X)^2 = \mu - \mu^2 = \mu(1 - \mu)
\]

**Problem N2-16**

\[
\text{var} \left( \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i^2 \text{var}(X_i)
\]

(the covariance terms are all zero if the variables are uncorrelated).

**Problem N2-17**

**Note:** There is no need to do this problem if you do N2-17 first. Both parts are special cases of the general formula derived in N2-17. Conversely, if you do this first, N2-17 can be done easily.

The first part:

\[
E(Z) = E \left( \frac{X - \mu}{\sigma} \right) = \frac{\mu - \mu}{\sigma} = 0
\]
and
\[ \text{var}(Z) = \text{var} \left( \frac{X - \mu}{\sigma} \right) = \frac{\sigma^2}{\sigma^2} = 1 \]

The second part:
\[ E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu + \sigma \cdot 0 = \mu \]

and
\[ \text{var}(X) = \text{var}(\mu + \sigma Z) = \sigma^2 \text{var}(Z) = \sigma^2 \cdot 1 = 1 \]

**Problem N2-18**

We need to solve the equations
\[
\begin{align*}
\mu_Y &= a + b\mu_X \\
\sigma_Y^2 &= b^2 \sigma_X^2
\end{align*}
\]

for \(a\) and \(b\). Solve the second and then plug into the first
\[
\begin{align*}
b &= \frac{\sigma_Y}{\sigma_X} \\
a &= \mu_Y - b\mu_X = \mu_Y - \frac{\sigma_Y}{\sigma_X} \mu_X
\end{align*}
\]

On the other hand, we could have used the solution to N2-17. First standardize, then “unstandardize”
\[
Y = \mu_Y + \sigma_Y Z = \mu_Y + \sigma_Y \frac{X - \mu_X}{\sigma_X} = \mu_Y + \frac{\sigma_Y}{\sigma_X} (X - \mu_X)
\]

which is the same solution as obtained by solving simultaneous equations.