What is the ARE of the two method of moments estimators compared on slides 61–62. deck 2.

These are $\bar{X}_n$ and $V_n$ considered as estimators of the mean of the Poisson distribution. The asymptotic distributions are

$$\bar{X}_n \approx N \left( \mu, \frac{\mu}{n} \right)$$

$$V_n \approx N \left( \mu, \frac{\mu^4 - \mu^2}{n} \right)$$

In order to figure out the asymptotic variance of the latter we need to calculate the fourth central moment of the Poisson distribution. We start with the moment generating function.

$$\varphi(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} \frac{e^{xt} \mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} \frac{(e^t \mu)^x}{x!} e^{-\mu}$$

$$= e^{\mu(e^t - 1)}$$
and this has derivatives

\[
\varphi'(t) = e^{\mu (e^t - 1)} \mu e^t \\
\varphi''(t) = e^{\mu (e^t - 1)} (\mu e^t)^2 + e^{\mu (e^t - 1)} \mu e^t \\
= e^{\mu (e^t - 1)} [\mu^2 e^{2t} + \mu e^t] \\
\varphi'''(t) = e^{\mu (e^t - 1)} [\mu^2 e^{2t} + \mu e^t] \mu e^t + e^{\mu (e^t - 1)} [2\mu^2 e^{2t} + \mu e^t] \\
= e^{\mu (e^t - 1)} [\mu^3 e^{3t} + 3\mu^2 e^{2t} + \mu e^t] \\
\varphi''''(t) = e^{\mu (e^t - 1)} [\mu^3 e^{3t} + 3\mu^2 e^{2t} + \mu e^t] \mu e^t \\
= e^{\mu (e^t - 1)} [\mu^4 e^{4t} + 6\mu^3 e^{3t} + 7\mu^2 e^{2t} + \mu e^t] \\
\]

and this gives ordinary moments

\[
\alpha_1 = E(X) = \varphi'(0) = \mu \\
\alpha_2 = E(X^2) = \varphi''(0) = \mu^2 + \mu \\
\alpha_3 = E(X^3) = \varphi'''(0) = \mu^3 + 3\mu^2 + \mu \\
\alpha_4 = E(X^4) = \varphi''''(0) = \mu^4 + 6\mu^3 + 7\mu^2 + \mu \\
\]

So, finally,

\[
\mu_4 = E\{ (X - \mu)^4 \} \\
= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4 \\
= \alpha_4 - 4\mu \alpha_3 + 6\mu^2 \alpha_2 - 4\mu^3 \alpha_1 + \mu^4 \\
= (\mu^4 + 6\mu^3 + 7\mu^2 + \mu) - 4\mu(\mu^3 + 3\mu^2 + \mu) + 6\mu^2(\mu^2 + \mu) - 4\mu^3 \mu + \mu^4 \\
= 3\mu^2 + \mu \\
\]

and the asymptotic variance of \( V_n \) is

\[
\mu_4 - \mu_2^2 = 3\mu^2 + \mu - \mu_2 = 2\mu^2 + \mu \\
\]

So \( \bar{X}_n \) has smaller asymptotic variance than \( V_n \) (for all values of \( \mu \)) and the ARE is

\[
\frac{\mu}{\mu + 2\mu^2} \\
\]

Note that the ARE goes to zero as \( \mu \) goes to infinity, so \( V_n \) gets arbitrarily bad for very large \( \mu \). Thus \( \bar{X}_n \) is not only the more obvious method of moments estimator but also the better one.