The Spacings of Record Values

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Abstract.

Let $\{R_i;\ i \geq 1\}$ be the sequence of record values generated from a sequence of i.i.d. continuous random variables, with distribution from the exponential family. In this paper we study the behavior of k-spacings of R_i, that is, $R_{k+i} - R_i$ for $i \geq 1$. We show that, under certain conditions, a normalized k-spacings empirically converges to a Gamma distribution. Counter examples show that the result is not valid when the conditions are violated. A strong law and a limiting distribution of the largest normalized spacings are also derived. In particular, these results conclude that the k-spacings go to infinity when the population distribution has heavy tail; the spacings go to zero when the tail is not heavy. Exact speeds of such convergence are obtained.

1 Introduction and main results

The record values was first studied by Chandler (1952). For full account of history and references, see, e.g., [1], [7] and [21]. A lot of properties on record values were understood: the joint distributions of record values was characterized, see, e.g., p.165 in [21]; the limiting distributions of records were proved to be the extreme value distributions, see, e.g., p. 174 in [21]; the extremal process, which is closely related to record values, was also studied, see p. 179 in [21], Deheuvels[8] and [9], and literatures therein; some limit theorems about record values are studied in Bose et al[4] and [5].

In this paper, we will investigate some properties of the spacings of record values. The study is inspired by similar research in random matrix theories (see, e.g., Gaudin[14], Mehta[20]), random graphs (see, e.g., Jacobson et al[16]), quantum chaos (see, e.g., Rudnick et al[22]), and number theories (see, e.g., Littlewood and Hardy[15], and Schmidt[23]). The main concern in this direction is the k-spacings of a given triangular array of random or non-random variables: $a_{n,1} \leq a_{n,2} \leq \cdots \leq a_{n,m}$ as n is large, where m depends on n. By k-spacings of this sequence we mean $a_{n,i+k} - a_{n,i}$ for $i = 1, 2, \cdots, m-k$. Two of the typical questions are the behavior of the empirical distributions of spacings and the largest spacings as n is large.

Now we state the main results of this paper.

Let $X, X_1, X_2, \cdots$ be i.i.d. random variables with cumulative distribution function $F(x)$ and

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Key Words: weak convergence, limiting distribution, Stein's Poisson approximation method, record value.

AMS (2000) subject classifications: 60F05, 60F15, 62E20, 60G07.

density function $p(x)$. That is,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} p(t)\,dt \tag{1.1}$$

for any $x \in \mathbb{R}$. Set

$$L_1 = 1,\ L_n = \inf\{k > L_{n-1};\ X_k > X_{L_{n-1}}\},\ n \ge 2. \tag{1.2}$$

Then $L_1, L_2, \cdots$ are called (upper) record times, and $X_{L_1}, X_{L_2}, X_{L_n}, \cdots$ are called (upper) record values. If the part "$X_k > X_{L_{n-1}}$" in (1.2) is replaced by "$X_k < X_{L_{n-1}}$", then the corresponding L_i's and X_{L_i}'s are referred to as lower record times and lower record values, respectively. Since the negative sign will change the lower case to the upper case, we will consider the upper case only in this paper. For convenience of notation, let $R_i = X_{L_i}$ for $i \ge 1$. We will use the following condition later:

There exists $A \in [-\infty, +\infty)$ such that $p(x) > 0$ for $x > A$, and $p(x) = 0$ for $x < A$ and $p(x)$ is continuous on $(A, +\infty)$. (1.3)

In particular, this implies that $F'(x) = p(x)$ for all $x > A$, and that the inverse of $F(x)$ exists on $(A, +\infty)$. Define $g(x) = p(F^{-1}(x)),\ 0 < x < 1$. The following condition will also be used:

There exist constants $\alpha > 0$ and $\beta > 0$, and a function $\omega(x)$ defined on $(0, +\infty)$ with $\lim_{x\to+\infty} \omega(x) = 0$ such that

$$g(1-t) = \beta t \left(\log \frac{1}{t}\right)^{(\alpha-1)/\alpha} \left\{1 + \omega\left(\frac{1}{t}\right)\right\} \tag{1.4}$$

as $t \to 0^+$. Throughout this paper, $\log x = \log_e x$ for $x > 0$.

This condition looks a bit strange at first sight. Proposition A.1 in Appendix says that (1.4) holds if the density function of random variable X is of the form $p(x) = c\, e^{-\kappa|x|^{\alpha} + b(x)} I\{x > A\}$, for some positive constants $c,\ \kappa$ and α, and some constant $A \ge -\infty$, and some function $b(x)$. Recall the parameter β in (1.4). When $b(x) \equiv 0$, we derive that $\beta = \alpha\kappa^{1/\alpha}$ and $\omega(x) = O((\log\log x)/\log x)$ as $x \to +\infty$; if $b(x)$ is roughly of order $o(x^{\alpha}/\log x)$ as $x \to +\infty$ (see the exact conditions in Proposition A.1), we obtain that $\beta = \alpha\kappa^{1/\alpha}$ and $\omega(x) = o(1/\log\log x)$ as $x \to +\infty$. This says that condition (1.4) holds for a great class of distributions in the exponential family.

The following result shows that the empirical distribution of the k-spacings of record values, suitably normalized, goes to a Gamma distribution.

THEOREM 1 *Suppose $p(x) = ce^{-x^{\alpha}} I(x > 0)$ for some constants $c > 0$ and $\alpha > 0$, or more generally, conditions (1.3) and (1.4) hold with $\omega(x) = o(1/\log(\log x))$ as $x \to +\infty$. Given integer $k \ge 1$. Let $D_i = i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)$ for $i \ge 1$ and $\mu_{n,k} = (1/n)\sum_{i=1}^{n} \delta_{D_i}$. Then, $\mu_{n,k}$ converges in distribution to μ with density $h(x) = \beta^k x^{k-1} e^{-\beta x} I(x \ge 0)/(k-1)!$ almost surely.*

REMARK 1.1 *Suppose X has the log-normal distribution, that is, $\log X \sim N(0, 1)$. Then $p(x)$ does not satisfy condition (1.4) because of Proposition A.2 in Appendix, yet the tail of this distribution is* *faster than the tail with the form of a rational function, for example,* $p(x) = x^{-2}I(x \geq 1)$*. Proposition A.3 says that Theorem 1 does not hold when* X *has log-normal distribution. However, the proposition also states that a different formulation of the spacings still gives a similar limiting distribution:* $(1/n)\sum_{i=1}^{n} \delta_{\tilde{D}_i}$ *converges to the Gamma distribution with density* $p(x) = 2^k x^{k-1} e^{-2x} I(x \geq 0)/(k-1)!$*, where* $\tilde{D}_i = \sqrt{i/2}(\log R_{k+i} - \log R_i)$ *for* $i \geq 1$*. This concludes that the spacings among records become larger than those as in Theorem 1.*

Another example is $F(x) = (1-(\log x)^{-1})I(x \geq e)$*. Then* $F^{-1}(x) = \exp(1/(1-x))$ *for* $x \in (0,1)$*. One can check from (1.5) below that*

$$\mathcal{L}(R_1, R_2, \cdots, R_n) = \mathcal{L}(e^{e^{S_1}}, e^{e^{S_2}}, \cdots, e^{e^{S_n}}).$$

where $S_i = \xi_1 + \xi_2 + \cdots + \xi_i$ *for* $i \geq 1$ *and* ξ_i*'s are i.i.d. random variables with distribution Exp(1). The spacings are even much larger than those as in the log-normal case.*

REMARK 1.2 *Roughly speaking, Theorem 1 says that the* k*-spacings of the record values multiplied by* $i^{(\alpha-1)/\alpha}$ *are random variables from a Gamma distribution. Recall that a typical distribution of* X *satisfying condition (1.4) is* $p(x) = ce^{-x^\alpha} I\{x > 0\}$ *for some* $c > 0$*. If* $\alpha = 1$*, the* k*-spacings without any normalization can be thought as a random sample from a Gamma distribution. If* $\alpha < 1$*, then* $i^{(\alpha-1)/\alpha} \to 0$*, which means that the spacings without normalization become larger and larger and go to infinity. If* $\alpha > 1$*, then* $i^{(\alpha-1)/\alpha} \to +\infty$*, which means that the spacings go to zero.*

The next result gives the scale of the largest normalized spacings.

THEOREM 2 *Suppose* $p(x) = ce^{-x^\alpha} I(x > 0)$ *for some constants* $c > 0$ *and* $\alpha > 0$*, or more generally, conditions (1.3) and (1.4) hold with* $\omega(x) = O(1/\log(\log x))$ *as* $x \to +\infty$*. Given integer* $k \geq 1$*, let* $W_n = \max_{1 \leq i \leq n}\{i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)\}$*. Then*

$$\lim_{n\to\infty} \frac{W_n}{\log n} \to \frac{1}{\beta} \quad a.s.$$

With the strong law above, we next give a refined result about W_n, which is the limiting distribution of the largest spacings.

THEOREM 3 *Suppose* $p(x) = ce^{-x^\alpha} I(x > 0)$ *for some constants* $c > 0$ *and* $\alpha > 0$*, or more generally, conditions (1.3) and (1.4) hold with* $\omega(x) = o(1/\log(\log x))$ *as* $x \to +\infty$*. Given integer* $k \geq 1$*, let* $W_n = \max_{1 \leq i \leq n}\{i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)\}$*. Then,*

$$P\left(W_n - \beta^{-1}\log n - \beta^{-1}(k-1)\log(\log n) \leq x\right) \to \exp\left(-\frac{1}{(k-1)!}e^{-\beta x}\right)$$

for any $x \in \mathbb{R}$.

The right hand side above is an extreme value distribution. The proofs of the above theorems rely on the following representation formula (see, e.g., Proposition 4.1 from [21]):

$$\begin{aligned} & \mathcal{L}(R_1, R_2, \cdots, R_n) \\ = \; & \mathcal{L}(F^{-1}(1-e^{-S_1}), F^{-1}(1-e^{-S_2}), \cdots, F^{-1}(1-e^{-S_n})) \end{aligned} \tag{1.5}$$

where F is the distribution of X as in (1.1), and $S_k = \xi_1 + \cdots + \xi_k$ for $k \geq 1$, and $\xi_1, \xi_2, \cdots$ are i.i.d.random variables with distribution $Exp(1)$.

The outline of this paper is as follows. We present all the proofs of theorems stated above in Section 2; some technical lemmas used in Section 2 and some tools on large deviations and Stein's Poisson approximation method are provided in Section 3.

2 Proofs

In this section, we provide the proofs of results stated in Section 1. We will use the following equivalent form of condition (1.4), which is convenient in discussion later.

There exist constants $\alpha > 0$ and $\beta > 0$, and a function $\omega(x)$ defined on $(0, +\infty)$ with $\lim_{x\to+\infty} \omega(x) = 0$ such that

$$\frac{t}{g(1-t)} \cdot \left(\log \frac{1}{t}\right)^{(\alpha-1)/\alpha} - \frac{1}{\beta} = \omega\left(\frac{1}{t}\right) \tag{2.1}$$

as $t \to 0^+$.

Proof of Theorem 1. By Theorem 11.3.3 of Dudley [12], it is enough to show that $\int_{\mathbb{R}} f(x)\mu_{n,k}(dx) \to \int_{\mathbb{R}} f(x)\mu(\, dx)$ as $n \to +\infty$ for any Lipschitz function $f(x)$ with Lipschitz norm equal to 1, in particular, $\|f\| := \sup_{x\in\mathbb{R}} |f(x)| \leq 1$. Now

$$\int_{\mathbb{R}} f(x)\mu_{n,k}(dx) = \frac{1}{n}\sum_{i=1}^{n} f\left(i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)\right).$$

We have to show that the right side above almost surely goes to

$$\frac{1}{(k-1)!}\int_0^{\infty} f(x)\beta^k x^{k-1}e^{-\beta x}\, dx = \frac{1}{(k-1)!}\int_0^{\infty} f\left(\frac{x}{\beta}\right) x^{k-1}e^{-x}\, dx.$$

Since $f(x)$ is bounded, we only need to prove that

$$\frac{1}{n}\sum_{i=[cn]}^{n} f\left(i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)\right) \to \frac{1-c}{(k-1)!}\int_0^{\infty} f\left(\frac{x}{\beta}\right) x^{k-1}e^{-x}\, dx \tag{2.2}$$

for any $c \in (0, 1)$. Recall formula (1.5), when we discuss quantities on $\{R_i;\ 1 \leq i \leq n\}$ for fixed n, we simply regard $R_i = F^{-1}(1 - e^{-S_i})$ for $1 \leq i \leq n$. By the Mean-value theorem, there exists $\theta_i \in [S_i, S_{k+i}]$ such that

$$i^{(\alpha-1)/\alpha}(R_{k+i} - R_i) = (S_{k+i} - S_i)\frac{e^{-\theta_i}}{g(1-e^{-\theta_i})} i^{(\alpha-1)/\alpha}. \tag{2.3}$$

Possibly there are many θ_i's satisfying the above equation. For definiteness, choose θ_i to be the infinimum of those to satisfy the above. By the continuity of $p(x)$ as assumed in condition (1.3), the equation (2.3) still holds for the new θ_i. Then such θ_i may not necessarily be measurable. But we do not need that, the condition $S_i \leq \theta_i \leq S_{k+i}$ is enough in later proofs. Define

$$H(x) = \frac{\beta x}{g(1-x)}\left(\log(x^{-1})\right)^{(\alpha-1)/\alpha},\ x > 0. \tag{2.4}$$

Then, by (2.3),

$$i^{(\alpha-1)/\alpha}(R_{k+i}-R_i)=\beta^{-1}(S_{k+i}-S_i)H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha}. \tag{2.5}$$

By condition (2.1), there exists $\delta\in(0,1)$ such that $|H(x)-1|\le 2\beta|\omega(x^{-1})|$ for all $0<x<\delta$. From (2.5) and the Lipschitz property of $f(x)$, we have that

$$\begin{aligned}&\left|f\left(i^{(\alpha-1)/\alpha}(R_{k+i}-R_i)\right)-f\left(\beta^{-1}(S_{k+i}-S_i)\right)\right|\\ \le\ &\beta^{-1}(S_{k+i}-S_i)\left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha}-1\right|.\end{aligned}$$

Obviously, $S_{k+i}-S_i\le k\cdot\max_{1\le j\le k+n}\xi_j$ for all $1\le i\le n$. To prove (2.2), it suffices to show that

$$\begin{aligned}\frac{1}{n}\sum_{i=[cn]}^{n}f\left(\beta^{-1}(S_{k+i}-S_i)\right)&\ \to\ (1-c)Ef\left(\beta^{-1}S_k\right)\\ &\ =\ \frac{1-c}{(k-1)!}\int_0^\infty f\left(\beta^{-1}x\right)x^{k-1}e^{-x}\,dx\quad a.s.\end{aligned} \tag{2.6}$$

and

$$\left(\max_{1\le i\le n+k}\xi_i\right)\cdot\max_{[cn]\le i\le n}\left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha}-1\right|\to 0\quad a.s. \tag{2.7}$$

To prove (2.6), it is enough to show that

$$\frac{1}{n}\sum_{i=[cn]}^{n}\left(f(S_{k+i}-S_i)-Ef(S_k)\right)\to 0\quad a.s.$$

Since $\sum_{i=[cn]}^{n}a_i=\sum_{i=1}^{n}a_i-\sum_{i=1}^{[cn]}a_i+a_{[cn]}$ for any $\{a_i;\ i\ge 1\}$, the above is equivalent to that

$$U_n:=\frac{1}{n}\sum_{i=1}^{[cn]}(f(S_{k+i}-S_i)-Ef(S_k))\to 0\quad a.s.$$

for any $c\in(0,1]$. Set $S_0=0$. Write

$$\begin{aligned}U_n&=\left(\frac{1}{n}\sum_{j=0}^{k-1}Z_{n,j}\right)-\frac{1}{n}\left(f(S_k)-Ef(S_k)\right)\quad\text{where}\\ Z_{n,j}&=\sum_{i=1}^{[([cn]-j)/k]+1}\{f(S_{ik+j}-S_{(i-1)k+j})-Ef(S_k)\}.\end{aligned}$$

Noting that the random variables in the second sum are bounded i.i.d. random variables, it is easy to check that $E(Z_{n,j})^4=O(n^2)$ as $n\to\infty$ for $0\le j\le k-1$. Then, since $\|f\|\le 1$, by the convexity of function $r(x)=x^4$,

$$\begin{aligned}E(U_n)^4&\le 8E\left(\frac{1}{n}\sum_{j=0}^{k-1}Z_{n,j}\right)^4+8\left(\frac{2}{n}\right)^4\\ &\le\frac{2^7}{n^4}+\frac{k^3}{n^4}\sum_{j=0}^{k-1}E(Z_{n,j})^4=O\left(\frac{1}{n^2}\right)\end{aligned}$$

as $n \to \infty$. Therefore,

$$\sum_{i=1}^{\infty} P(|U_n| \geq \epsilon) \leq \epsilon^{-4} \sum_{n=1}^{\infty} E(U_n)^4 < \infty.$$

By the Borel-Cantelli lemma, $U_n \to 0$ *a.s.* Thus (2.6) follows.

Now $P(\max_{1\leq i\leq n} \xi_i \geq 3\log n) \leq nP(\xi_1 > 3\log n) = 1/n^2$. Then $\sum_{n=1}^{\infty} P(\max_{1\leq i\leq n} \xi_i \geq 3\log n) < \infty$. By the Borel-Cantelli lemma again,

$$\limsup_{n\to\infty} \frac{\max_{1\leq i\leq n} \xi_i}{\log n} \leq 3 \quad a.s. \tag{2.8}$$

Note that $|ab-1| \leq |a-1|\cdot|b-1| + |a-1| + |b-1|$ for any $a, b \in \mathbb{R}$. Thus, to prove (2.7), it suffices to show that

$$(\log n) \cdot \max_{[cn]\leq i\leq n} |H(e^{-\theta_i}) - 1| \to 0 \quad a.s. \tag{2.9}$$

$$(\log n) \cdot \max_{[cn]\leq i\leq n} \left| \left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1 \right| \to 0 \quad a.s. \tag{2.10}$$

Given $x > 0$ such that $|x-1| < 1/2$. By the Mean-value theorem, there exists ξ between 1 and x such that

$$\left| x^{(\alpha-1)/\alpha} - 1 \right| = |x-1| \cdot \xi^{-1/\alpha} \cdot \frac{|\alpha-1|}{\alpha} \leq \frac{2^{1/\alpha}|\alpha-1|}{\alpha} \cdot |x-1|. \tag{2.11}$$

provided $|x-1| < 1/2$. Thus, to prove (2.10) it is enough to show

$$(\log n) \cdot \max_{[cn]\leq i\leq n} \left| \frac{i}{\theta_i} - 1 \right| \to 0 \quad a.s. \tag{2.12}$$

First, since $S_i \leq \theta_i \leq S_{k+i}$,

$$\begin{aligned} \max_{[cn]\leq i\leq n} \left| \frac{i}{\theta_i} - 1 \right| &\leq \frac{1}{S_{[cn]}} \cdot \max_{[cn]\leq i\leq n} |\theta_i - i| \\ &\leq \frac{1}{S_{[cn]}} \cdot \max_{[cn]\leq i\leq n+k} (k + |S_i - i|). \end{aligned}$$

By the Kolmogorov's strong Law of Large Numbers, $S_{[cn]}/n \to c$ *a.s.* as $n \to \infty$. Therefore, to see (2.12), it is enough to prove that $n^{-1}(\log n)\max_{[cn]\leq i\leq n+k} |S_i - i| \to 0$ *a.s.* This is obvious, since $\limsup_{i\to\infty} |S_i - i|/\sqrt{2i\log\log i} < \infty$ *a.s.* by Hartman-Wintner's Law of Iterated Logarithm, then

$$\begin{aligned} \frac{\max_{[cn]\leq i\leq n+k} |S_i - i|}{n/\log n} &\leq \frac{(\log n)^{3/2}}{\sqrt{n}} \max_{[cn]\leq i\leq n+k} \frac{|S_i - i|}{\sqrt{2i\log\log i}} \\ &\leq \frac{(\log n)^{3/2}}{\sqrt{n}} \sup_{i\geq[cn]} \frac{|S_i - i|}{\sqrt{2i\log\log i}} \to 0 \quad a.s. \end{aligned}$$

as $n \to \infty$ since the supremum goes to 1 a.s. We obtain (2.12).

Finally, let's prove (2.9). By condition (2.1) and the condition that $\omega(x) = o(1/\log\log x)$ as $x \to +\infty$, for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that

$$\left| \frac{\beta t}{g(1-t)} \left(\log\left(t^{-1}\right)\right)^{(\alpha-1)/\alpha} - 1 \right| \leq \frac{\epsilon}{\log\log(t^{-1})}$$

as $0 < t < \delta$. By the Law of Large Numbers, $\lim_{i\to+\infty} S_i = +\infty$ *a.s.*, therefore $\max_{[cn]\le i\le n} e^{-\theta_i} \le e^{-S_{[cn]}} \to 0$ *a.s.* as $n \to +\infty$. Recalling (2.4), we have that, with probability one,

$$(\log n) \max_{[cn]\le i\le n} |H(e^{-\theta_i}) - 1| \le \frac{\epsilon \log n}{\log S_{[cn]}} \to \epsilon \quad a.s.$$

as $n \to +\infty$ by the Law of Large Numbers again. Then (2.2) follows. ■

We need the following lemma to prove Theorem 3.

LEMMA 2.1 *Suppose the conditions in Theorem 3 hold. Recall $D_i = i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)$ for $i \ge 1$. Then $P(\max_{1\le i\le\sqrt{n}} D_i \ge \beta^{-1}(\log n)) \to 0$ as $n \to \infty$.*

Proof. *Step 1.* Let $\alpha' = \max\{0, (\alpha-1)/\alpha\}$ and $\rho = F(\beta^{-1}(\log n)^{1-\alpha'})$, where $F(x)$ is the cumulative distribution function of X. By condition (1.3) and Proposition A.2 in Appendix, there exists constants $r_2 > r_1 > 0$ and $\varsigma > 0$ such that $r_1(\log n)^{\varsigma} < -\log(1-\rho) < r_2(\log n)^{\varsigma}$ as n is sufficiently large. Set

$$m_n = \min\left\{[\log n],\ \left[-\frac{1}{2}\log(1-\rho)\right]\right\}$$

for all $n \ge 1$. Then $m_n \to +\infty$ as $n \to \infty$. Note that

$$\max_{1\le i\le m_n} D_i \le m_n^{\alpha'} R_{k+m_n} \le (\log n)^{\alpha'} R_{k+m_n}.$$

Then, by the fact that $\mathcal{L}(R_i) = \mathcal{L}(F^{-1}(1-e^{-S_i}))$ for all $i \ge 1$.

$$\begin{aligned} P\left(\max_{1\le i\le m_n} D_i \ge \beta^{-1}\log n\right) &\le P\left(R_{k+m_n} \ge \beta^{-1}(\log n)^{1-\alpha'}\right) \\ &\le P\left(S_{k+m_n} \ge -\log\left\{1 - F\left(\beta^{-1}(\log n)^{1-\alpha'}\right)\right\}\right), \end{aligned}$$

which is bounded by $P(S_{k+m_n} \ge 2m_n)$ by the definition of m_n. This probability goes to zero by the weak Law of Large Numbers. Hence, to prove the lemma, it suffices to show that

$$P\left(\max_{m_n\le i\le\sqrt{n}} D_i \ge \beta^{-1}\log n\right) \to 0 \tag{2.13}$$

as $n \to \infty$.

Step 2. We now prove (2.13). Notice

$$P\left(\max_{m_n\le i\le\sqrt{n}} D_i \ge \beta^{-1}\log n\right) \le \sum_{m_n\le i\le\sqrt{n}} P\left(D_i \ge \beta^{-1}\log n\right) \tag{2.14}$$

As in (2.3) and (2.5), there exists $\theta_i \in [S_i, S_{i+k}]$ such that

$$D_i = \beta^{-1}(S_{k+i} - S_i)H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} \tag{2.15}$$

where $H(x)$ is as in (2.4). Trivially, $|x-1| \leq \max\{|a-1|, |b-1|\}$ if $a \leq x \leq b$ for any $a, b \in \mathbb{R}$. Thus, if $|(\theta_i/i)-1| \geq (2\log i)/\sqrt{i}$, then either $|S_i/i - 1| \geq (2\log i)/\sqrt{i}$ or

$$\begin{aligned}\frac{2\log i}{\sqrt{i}} \leq \left|\frac{S_{k+i}}{i} - 1\right| &\leq \left|\frac{S_{k+i}-(k+i)}{i}\right| + \frac{k}{i} \\ &\leq \frac{k+i}{i} \cdot \left|\frac{S_{k+i}-(k+i)}{k+i}\right| + \frac{k}{i},\end{aligned}$$

which in turn implies that $|S_{k+i}/(k+i) - 1| \geq (\log(k+i))/\sqrt{k+i}$ as i is sufficiently large. Thus, noting that $E\xi_1 = 1$ and $\mathrm{Var}(\xi_1) = 1$, by (ii) of Lemma A.2 ,

$$P\left(\left|\frac{\theta_i}{i} - 1\right| \geq \frac{2\log i}{\sqrt{i}}\right) \leq 2 \cdot \max_{i \leq j \leq k+i} P\left(\left|\frac{S_j}{j} - 1\right| \geq \frac{\log j}{\sqrt{j}}\right) \leq e^{-(\log i)^2/3} \tag{2.16}$$

as i is sufficiently large. By condition (2.1), there exists $\delta \in (0,1)$ and $K \in (0, +\infty)$ such that $|H(x)-1| \leq K|\omega(x^{-1})|$ as $0 < x < \delta$. Obviously, if $\epsilon \in (0, 1/2)$ and $|1-x| < \epsilon$, we have that $|1-x^{-1}| \leq 2|x-1|$. Thus, if $|(\theta_i/i)-1| \leq (\log i)/\sqrt{i}$, then $\theta_i \geq i - \sqrt{i}\log i$ and $|(i/\theta_i)-1| \leq 2(\log i)/\sqrt{i}$ as i is sufficiently large, use (2.11) and the inequality that $|ab-1| \leq |a-1|\cdot|b-1| + |a-1| + |b-1|$ to get

$$\begin{aligned}& \left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right| \\ \leq\ & K \cdot K_\alpha \cdot |\omega(e^{\theta_i})| \cdot \left|\frac{i}{\theta_i} - 1\right| + K|\omega(e^{\theta_i})| + K_\alpha \left|\frac{i}{\theta_i} - 1\right| \\ =\ & o\left(\frac{1}{\log\log e^{\theta_i}}\right) \cdot O\left(\frac{\log i}{\sqrt{i}}\right) + o\left(\frac{1}{\log\log e^{\theta_i}}\right) + O\left(\frac{\log i}{\sqrt{i}}\right) = o\left(\frac{1}{\log i}\right) \ a.s.\end{aligned}$$

for all $i \geq m_n$ and n is sufficiently large, where $K_\alpha = 2^{1/\alpha}|\alpha-1|/\alpha$. We use the fact that $\omega(x)\log(\log x) \to 0$ as $x \to +\infty$ in the first equality above. By (2.16),

$$P\left(\left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right| \geq \frac{1}{\log i}\right) \leq e^{-(\log i)^2/3}$$

for all $i \geq m_n$ as n is sufficiently large. From (2.14), (2.15) and the above, we have that

$$\begin{aligned}& P\left(D_i > \beta^{-1}\log n\right) \\ \leq\ & P\left(S_k \geq (\log n)\left(1 + \frac{1}{\log m_n}\right)^{-1}\right) + e^{-(\log i)^2/3}\end{aligned} \tag{2.17}$$

for all $i \geq m_n$ as n is sufficiently large. By L'Hospital's rule, one can easily check that

$$P(S_k > t) = \int_t^\infty \frac{x^{k-1}e^{-x}}{(k-1)!}\,dx \sim \frac{t^{k-1}e^{-t}}{(k-1)!} \tag{2.18}$$

as $t \to +\infty$. Recalling (2.14), we obtain that

$$\begin{aligned}& P\left(\max_{m_n \leq i \leq \sqrt{n}} D_i \geq \beta^{-1}\log n\right) \\ \leq\ & 2\sqrt{n}\cdot(\log n)^{k-1}e^{-(\log n)(1+o(1))} + \sum_{m_n \leq i \leq \sqrt{n}} e^{-(\log i)^2/3}\end{aligned}$$

as n is sufficiently large. The middle term above goes to zero evidently; since $m_n \to +\infty$ and $\int_3^{+\infty} e^{-(\log x)^2/3}\, dx < \infty$, the last term is bounded by

$$e^{-(\log m_n)^2/3} + \int_{m_n}^{\infty} e^{-(\log x)^2/3}\, dx \to 0$$

as $n \to +\infty$. We finally get (2.13). ■

Proof of Theorem 3. By Lemma 2.1, it is enough to prove that

$$P(\max_{[\sqrt{n}]\le i\le n} (\beta D_i) \ge \log n + (k-1)\log(\log n) + \beta x) \to 1 - \exp\left(-\frac{1}{(k-1)!}e^{-\beta x}\right) \tag{2.19}$$

as $n \to \infty$. From (2.15),

$$\beta D_i = (S_{k+i} - S_i) + (S_{k+i} - S_i)\left\{H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right\}.$$

By triangle inequality,

$$\begin{aligned}
& |\max_{[\sqrt{n}]\le i\le n} (\beta D_i) - \max_{[\sqrt{n}]\le i\le n} (S_{k+i} - S_i)| \\
\le\ & k(\max_{1\le i\le n+k} \xi_i)\cdot \max_{[\sqrt{n}]\le i\le n}\left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right|.
\end{aligned}$$

From Lemma A.4,

$$\begin{aligned}
& P(\max_{[\sqrt{n}]\le i\le n} (S_{k+i} - S_i) \ge \log n + (k-1)(\log(\log n) + \beta x) \\
=\ & P(\max_{1\le i\le r_n} (S_{k+i} - S_i) \ge \log r_n + (k-1)(\log(\log r_n) + \beta x + o(1)) \\
\to\ & 1 - \exp\left(-\frac{1}{(k-1)!}e^{-\beta x}\right)
\end{aligned}$$

as $n \to \infty$, where $r_n = n - [\sqrt{n}] + 1$. Second, it is known that $(\max_{1\le i\le n} \xi_i)/\log n \to 1$ in probability. Thus, to prove the theorem, we only need to prove

$$(\log n)\cdot \max_{[\sqrt{n}]\le i\le n}\left|H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right| \to 0 \tag{2.20}$$

in probability. Again, recall that $|ab-1| \le |a-1|\cdot|b-1| + |a-1| + |b-1|$ for any $a, b \in \mathbb{R}$. As in the argument between (2.10) and (2.12), to show (2.20), it suffices to demonstrate that

$$(\log n)\cdot \max_{[\sqrt{n}]\le i\le n} |H(e^{-\theta_i}) - 1| \to 0 \ \text{ in probability;} \tag{2.21}$$

$$(\log n)\cdot \max_{[\sqrt{n}]\le i\le n} \left|\frac{i}{\theta_i} - 1\right| \to 0 \ \text{ in probability} \tag{2.22}$$

as $n \to \infty$. By (2.16),

$$P\left(\left|\frac{\theta_i}{i} - 1\right| \ge \frac{2\log i}{\sqrt{i}}\right) \le e^{-(\log i)^2/3}$$

as i is sufficiently large. Obviously, if $|x-1|<\delta$ and $\delta<1/2$, then $|1-x^{-1}|<2\delta$. Therefore,

$$\begin{aligned} P\left(\left|\frac{i}{\theta_i}-1\right| \geq \frac{4\log i}{\sqrt{i}}\right) &\leq P\left(\left|\frac{\theta_i}{i}-1\right| \geq \frac{2\log i}{\sqrt{i}}\right) \\ &\leq e^{-(\log i)^2/3} \end{aligned} \tag{2.23}$$

as i is sufficiently large. If $|(i/\theta_i)-1| \leq 4(\log i)/\sqrt{i}$ for all $[\sqrt{n}] \leq i \leq n$, then

$$(\log n)\cdot \max_{[\sqrt{n}]\leq i\leq n}\left|\frac{i}{\theta_i}-1\right| \leq \frac{5(\log n)^2}{n^{1/4}} < \epsilon \tag{2.24}$$

for any $\epsilon>0$ when n is sufficiently large. It follows that

$$\begin{aligned} P\left((\log n)\cdot \max_{[\sqrt{n}]\leq i\leq n}\left|\frac{i}{\theta_i}-1\right| \geq \epsilon\right) &\leq n\cdot \max_{[\sqrt{n}]\leq i\leq n} P\left(\left|\frac{i}{\theta_i}-1\right| \geq \frac{4\log i}{\sqrt{i}}\right) \\ &\leq ne^{-(\log n)^2/13} \end{aligned}$$

as n is sufficiently large. So (2.22) follows. Now we prove (2.21). By condition (2.1), there exists $K>0$ and $\delta\in(0,1)$ such that $|H(x)-1| \leq K|\omega(x^{-1})|$ as $0<x<\delta$. Thus, if $|(\theta_i/i)-1| \leq 4(\log i)/\sqrt{i}$ for all $[\sqrt{n}] \leq i \leq n$, then $\theta_i \geq i-4\sqrt{i}\log i \geq \sqrt{n}/2$ for all $[\sqrt{n}] \leq i \leq n$ as n is sufficiently large. Therefore,

$$\begin{aligned} & (\log n)\cdot \max_{[\sqrt{n}]\leq i\leq n} |H(e^{-\theta_i})-1| \\ \leq\ & (K\log n)\cdot \max_{[\sqrt{n}]\leq i\leq n} \frac{1}{\log(\log e^{\theta_i})}\cdot\left(\max_{[\sqrt{n}]\leq i\leq n}\{|\omega(e^{\theta_i})|\log(\log e^{\theta_i})\}\right) \\ \leq\ & (3K)\sup_{x\geq e^{\sqrt{n}/2}}\{|\omega(x)|\log(\log x)\} \to 0 \end{aligned}$$

as n goes to $+\infty$ by the given condition. Finally, by (2.23),

$$\begin{aligned} & P\left((\log n)\cdot \max_{[\sqrt{n}]\leq i\leq n}|H(e^{-\theta_i})-1| > \epsilon\right) \\ \leq\ & \sum_{[\sqrt{n}]\leq i\leq n} P\left(\left|\frac{i}{\theta_i}-1\right| > \frac{4\log i}{\sqrt{i}}\right) \\ \leq\ & ne^{-(\log[\sqrt{n}])^2/3} \to 0 \end{aligned}$$

as $n\to+\infty$. The assertion (2.21) is yielded. ∎

Proof of Theorem 2. We will next show that

$$\limsup_{n\to\infty}\frac{W_n}{\log n} \leq \frac{1}{\beta}\quad a.s. \tag{2.25}$$

$$\liminf_{n\to\infty}\frac{W_n}{\log n} \geq \frac{1}{\beta}\quad a.s. \tag{2.26}$$

Upper bound. Given $\epsilon>0$. Let $\gamma=(1/2)\min\{\alpha,1\}$. We claim that

$$P\left(U_n \geq (1+\epsilon)\beta^{-1}\log n\right) \leq e^{-(\log n)^{\gamma}} \tag{2.27}$$

as n is sufficiently large, where $U_n = \max_{1\le i\le k_n}\{i^{(\alpha-1)/\alpha}D_i\}$ and $k_n = [(\log n)^\gamma]$. Note that $\max_{1\le i\le k_n}\{i^{(\alpha-1)/\alpha}D_i\} \le (\log n)^\tau F^{-1}(1-e^{-S_{k_n+k}})$, where $\tau = \max\{(\alpha-1)\gamma/\alpha, 0\}$. Obviously, $1-\tau \ge 2\gamma/\alpha$. Then, for $n \ge e$,

$$
\begin{aligned}
& P\left(\max_{1\le i\le k_n}\{i^{(\alpha-1)/\alpha}D_i\} \ge (1+\epsilon)\beta^{-1}\log n\right) \\
\le\ & P\left(F^{-1}(1-e^{-S_{k_n+k}}) \ge (1+\epsilon)\beta^{-1}(\log n)^{2\gamma/\alpha}\right) \\
\le\ & P\left(S_{k_n+k} \ge -\log(1-F((1+\epsilon)\beta^{-1}(\log n)^{2\gamma/\alpha}))\right)
\end{aligned}
$$

By condition (1.4) and Proposition A.2, there exists constant $K_1 > 0$ such that $1 - F((1+\epsilon)\beta^{-1}(\log n)^{2\gamma/\alpha}) \le \exp(-K_1(\log n)^{2\gamma})$ as n is sufficiently large. Taking $-\log$ for both sides, we have from (i) of Lemma A.2 that the last probability is bounded by $P(S_{k_n+k}/(k_n+k) \ge e^2) \le 2e^{-I(e^2)k_n}$. It is easy to check that $I(x) = x - \log x - 1$ for $x > 0$. Claim (2.27) then follows since $I(e^2) \ge 4$.

Now, let $V_n = \max_{k_n\le i\le n} D_i$, where $D_i = i^{(\alpha-1)/\alpha}(R_{k+i} - R_i)$. For any $\epsilon > 0$,

$$
P\left(V_n \ge (1+2\epsilon)\beta^{-1}\log n\right) \le \sum_{i=k_n}^{n} P\left(\beta D_i \ge (1+2\epsilon)\log n\right). \tag{2.28}
$$

From (2.3), there exists $\theta_i \in [S_i, S_{i+k}]$ such that

$$
D_i = \beta^{-1}(S_{k+i} - S_i)H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha}
$$

where $H(x)$ is as in (2.4). Write

$$
\beta D_i = (S_{k+i} - S_i) + (S_{k+i} - S_i)\left(H(e^{-\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right). \tag{2.29}
$$

Then, by the inequality $|ab-1| \le |a-1|\cdot|b-1| + |a-1| + |b-1|$, we have that

$$
\left|H(e^{\theta_i})\left(\frac{i}{\theta_i}\right)^{(\alpha-1)/\alpha} - 1\right| \quad \le \quad B_i\cdot C_i + B_i + C_i \tag{2.30}
$$

where $B_i = |H(e^{-\theta_i}) - 1|$ and $C_i = |\,(i/\theta_i)^{(\alpha-1)/\alpha} - 1|$. Thus

$$
\begin{aligned}
& P(\beta D_i \ge (1+2\epsilon)\log n) \\
\le\ & P(S_k \ge (1+\epsilon)\log n) + P((S_{k+i} - S_i)(B_i\cdot C_i + B_i + C_i) \ge \epsilon\log n)
\end{aligned} \tag{2.31}
$$

for any $k_n \le i \le n$. We claim that there exists a constant $K_2 > 0$ such that

$$
P\left(B_i \ge \frac{K_2}{\log i}\right) \le e^{-(\log i)^2/3} \text{ and } P\left(C_i > \frac{K_2\log i}{\sqrt{i}}\right) \le e^{-(\log i)^2/3} \tag{2.32}
$$

as i sufficiently large. Indeed, as in (2.23), we have that

$$
P\left(|\frac{i}{\theta_i} - 1| \ge \frac{4\log i}{\sqrt{i}}\right) \le e^{-(\log i)^2/3} \tag{2.33}
$$

as i is sufficiently large. Reviewing the argument in (2.11), $|(i/\theta_i)^{(\alpha-1)/\alpha} - 1| \le K_3|(i/\theta_i) - 1| \le 4K_3(\log i)/\sqrt{i}$ if $|(i/\theta_i) - 1| \le 4(\log i)/\sqrt{i}$ and i is sufficiently large, where $K_3 = 2^\alpha|1 - \alpha^{-1}|$. Therefore

$$P\left(C_i > \frac{4K_3 \log i}{\sqrt{i}}\right) \le e^{-(\log i)^2/3} \tag{2.34}$$

as i is sufficiently large. So the second inequality of (2.32) follows for any $K_2 \ge 4K_3$. Now, by (2.16), $P(|(\theta_i/i) - 1| \ge 2(\log i)/\sqrt{i}) \le e^{-(\log i)^2/3}$ if i is large enough. If $|(\theta_i/i) - 1| \le 2(\log i)/\sqrt{i}$ and i is large, by condition (2.1) and the assumption on $\omega(x)$, there exists a constant $K_4 > 0$ such that $B_i \le K_4(\log(\log e^{\theta_i}))^{-1} \le 2K_4/\log i$ as i is sufficiently large. It follows that

$$P\left(B_i \ge \frac{3K_4}{\log i}\right) \le e^{-(\log i)^2/3} \tag{2.35}$$

as i is large enough. Now taking $K_2 = 4K_3 + 3K_4$, we get the first inequality of (2.32).

Now, if $B_i \le K_2/\log i$ and $C_i \le (K_2 \log i)/\sqrt{i}$, then $B_i \cdot C_i + B_i + C_i \le K_5/\log i$ for some constant $K_5 > 0$ as i is sufficiently large. Therefore

$$P\left(B_i \cdot C_i + B_i + C_i > \frac{K_5}{\log i}\right) \le 2e^{-(\log i)^2/3} \tag{2.36}$$

as i is sufficiently large. Then, for $k_n \le i \le n$, noting $(\epsilon/K_5)(\log n)\log k_n) > (1+\epsilon)\log n$ as n is sufficiently large,

$$\begin{aligned} & P((S_{k+i} - S_i)(B_i \cdot C_i + B_i + C_i) \ge \epsilon \log n) \\ \le \quad & P\left(S_k \ge (1+\epsilon)\log n\right) + 2e^{-(\log i)^2/3} \end{aligned} \tag{2.37}$$

as n is large enough. From (2.18), $P(S_k > (1+\epsilon)\log n) \le n^{-1-(\epsilon/2)}$ as n is large enough. Combining (2.31) and (2.37), we get

$$P(\beta D_i \ge (1+2\epsilon)\log n) \le \frac{3}{n^{1+(\epsilon/2)}} + 2e^{-(\log i)^2/3}$$

for all $k_n \le i \le n$ and large n. By (2.28),

$$P\left(V_n \ge (1+2\epsilon)\beta^{-1}\log n\right) \le \frac{3}{n^{\epsilon/2}} + 2\sum_{i=k_n}^{n} e^{-(\log i)^2/3}$$

as n is sufficiently large. The last sum is dominated by

$$\begin{aligned} & e^{-(\log k_n)^2/3} + \sum_{i=k_n+1}^{n} \int_{i-1}^{i} e^{-(\log x)^2/3}\,dx \\ \le \quad & e^{-(\log k_n)^2/3} + \int_{k_n}^{\infty} e^{-(\log x)^2/3}\,dx. \end{aligned}$$

Use $(\log x)^2 \ge (\log k_n)\log x$ for $x \ge k_n$ to obtain that

$$\int_{k_n}^{\infty} e^{-(\log x)^2/3}\,dx \le \int_{k_n}^{\infty} x^{-(\log k_n)/3}\,dx = \left(\frac{\log k_n}{3} - 1\right)^{-1} (k_n)^{1-(\log k_n)/3},$$

which is bounded by $e^{-(\log k_n)^2/4}$ as n is sufficiently large. Combing all the above, keep in mind that $k_n = [(\log n)^\gamma]$, we have

$$P\left(V_n \geq (1+2\epsilon)\beta^{-1}\log n\right) \leq 5e^{-(\log k_n)^2/4} \leq e^{-(\log\log n)^2/5}$$

as n is large enough. Noting $W_n \leq \max\{U_n, V_n\}$. The last inequality together with (2.27) concludes that

$$P\left(\frac{W_n}{\log n} \geq (1+2\epsilon)\beta^{-1}\right) \leq 2e^{-(\log\log n)^2/5}$$

as n is sufficiently large. Take $p_n = [e^n]+1$ for all $n \geq 1$. Then $\sum_{n\geq 2} P(W_{p_n}/\log p_n \geq (1+2\epsilon)\beta^{-1}) \leq 2\sum_{n\geq 2} e^{-(\log n)^2/5} < \infty$. By the Borel-Cantelli Lemma,

$$\limsup_{n\to\infty} \frac{W_{p_n}}{\log p_n} \leq (1+2\epsilon)\beta^{-1} \quad a.s.$$

as $n \to \infty$. Observe that W_n is non-decreasing. For any $k \geq 3$, there exists $n \geq 1$ such that $p_n \leq k < p_{n+1}$. Then $W_k/\log k \leq (W_{p_{n+1}}/\log p_{n+1}) \cdot (\log p_{n+1})/(\log p_n)$ for all $p_n \leq k < p_{n+1}$. Since $(\log p_{n+1})/(\log p_n) \to 1$ as $n \to \infty$, we eventually get

$$\limsup_{k\to\infty} \frac{W_k}{\log k} \leq (1+2\epsilon)\beta^{-1} \quad a.s.$$

for any $\epsilon > 0$. This gives (2.25).

Lower bound. By (2.29) and (2.30),

$$S_{k+i} - S_i \leq \beta D_i + (S_{k+i} - S_i)(B_i \cdot C_i + B_i + C_i))$$

for any $i \geq 1$. Set $\gamma_n = \max_{\sqrt{n}\leq i\leq n}(B_iC_i + B_i + C_i)$. Then

$$\max_{\sqrt{n}\leq i\leq n}(S_{k+i} - S_i) \tag{2.38}$$
$$\leq \max_{\sqrt{n}\leq i\leq n}(\beta D_i) + \max_{\sqrt{n}\leq i\leq n}(S_{k+i} - S_i)\cdot\gamma_n \tag{2.39}$$

which gives $\max_{\sqrt{n}\leq i\leq n}(\beta D_i) \geq (1-\gamma_n)\max_{\sqrt{n}\leq i\leq n}(S_{k+i} - S_i)$. By (2.36),

$$\begin{aligned} P\left(\gamma_n \geq \frac{K_5}{\log\sqrt{n}}\right) &\leq n\cdot \max_{\sqrt{n}\leq i\leq n} P\left(B_iC_i + B_i + C_i \geq \frac{K_5}{\log i}\right) \\ &\leq ne^{-(\log n)^2/13} \end{aligned} \tag{2.40}$$

as n is sufficiently large. Thus, for any $\epsilon \in (0, 1/4)$,

$$\begin{aligned} P\left(W_n \leq (1-2\epsilon)\beta^{-1}\log n\right) &\leq P\left(\max_{\sqrt{n}\leq i\leq n}(\beta D_i) \leq (1-2\epsilon)\log n\right) \\ &\leq P\left((1-\gamma_n)\max_{\sqrt{n}\leq i\leq n}(S_{k+i} - S_i) \leq (1-2\epsilon)\log n\right). \end{aligned}$$

Considering the event in the last probability by distinguishing $\gamma_n \geq K_5/\log\sqrt{n}$ or not, and noticing $(1-2\epsilon)(1-(K_5/\log\sqrt{n}))^{-1} \to 1-2\epsilon$ as $n \to \infty$. By (2.40),

$$\begin{aligned} &P\left(W_n \leq (1-2\epsilon)\beta^{-1}\log n\right) \\ \leq\ & ne^{-(\log n)^2/13} + P\left(\max_{\sqrt{n}\leq i\leq n}(S_{k+i} - S_i) \leq (1-\epsilon)\log n\right) \end{aligned} \tag{2.41}$$

as n is sufficiently large. Since $\{\xi_i;\ i \geq 1\}$ are i.i.d. random variables,

$$\begin{aligned} & P\left(\max_{\sqrt{n}\leq i\leq n}(S_{k+i}-S_i)\leq (1-\epsilon)\log n\right) \\ \leq\ & P\left(\max_{1\leq i\leq n-[\sqrt{n}]+1}(S_{k+i}-S_i)\leq (1-\epsilon)\log n\right) \\ \leq\ & P\left(\max_{1\leq j\leq q_n}(S_{(j+1)k}-S_{jk})\leq (1-\epsilon)\log n\right) \end{aligned}$$

where $q_n = [(n-\sqrt{n}+1)/k]-1$. By independence, the above probability is identical to

$$(1-P(S_k > (1-\epsilon)\log n))^{q_n} \leq e^{-q_n P(S_k>(1-\epsilon)\log n)}.$$

From (2.18),

$$P(S_k > (1-\epsilon)\log n) \sim \frac{(1-\epsilon)^{k-1}}{(k-1)!}\cdot\frac{(\log n)^{k-1}}{n^{1-\epsilon}}$$

as $n\to\infty$. So there exists a constant $K_6>0$ depending on k only such that $q_nP(S_k > (1-\epsilon)\log n) \geq K_6 n^{\epsilon}$ for n large enough. Collecting all the above, we have

$$P\left(W_n \leq (1-2\epsilon)\beta^{-1}\log n\right) \leq ne^{-(\log n)^2/13} + e^{-K_6 n^{\epsilon}}$$

as n is sufficiently large. Evidently, the sum of the right hand side over all $n\geq 2$ is finite. By the Borel-Cantelli lemma,

$$\liminf_{n\to\infty}\frac{W_n}{\log n} \geq (1-2\epsilon)\beta^{-1} \quad a.s.$$

for any $\epsilon \in (0, 1/4)$. Letting $\epsilon \to 0^+$, we finally obtain (2.26). ■

3 Appendix

In this section, we first show that condition (1.4) holds for a great class of probability distributions. Then we provide an example that Theorem 1 does not hold when condition (1.4) is violated. At last we collect some tools and technical results used in Section 2.

Proposition A.1 *Let $\kappa>0,\ c>0,\ \alpha>0$ and $A\in[-\infty,+\infty)$ be constants. Let $b(x)$, $x>A$, be a function such that the density function of X is $p(x) = ce^{-\kappa|x|^{\alpha}+b(x)}I\{x>A\}$. The following hold.*

(i) If $b(x)=0$ for all $x\geq A$, then condition (1.4) holds with $\beta=\alpha\kappa^{1/\alpha}$ and $\omega(x) = O((\log\log x)/\log x)$ as $x\to+\infty$.

(ii) If $b(x)$ is a twice differentiable function such that $b'(x) = O(x^{\alpha-1}/\log x)$ and $b''(x) = o(x^{\alpha-2})$ as $x\to+\infty$, then condition (1.4) holds with $\beta=\alpha\kappa^{1/\alpha}$ and $\omega(x) = O(1/\log\log x)$ as $x\to+\infty$. If, further, $b'(x) = o(x^{\alpha-1}/\log x)$ as $x\to+\infty$, then $\beta=\alpha\kappa^{1/\alpha}$ and $\omega(x) = o(1/\log\log x)$ as $x\to+\infty$.

The next result is a weak converse of condition (1.4).

Proposition A.2 *Suppose conditions (1.3) and (1.4) hold. Then, for any $r > 1$ and $s \in (0,1)$, $e^{-r\alpha x^\alpha/\beta} \le 1 - F(x) \le e^{-s\alpha x^\alpha/\beta}$ as x is sufficiently large.*

Now we start to prove the above two propositions. The following lemma will be used for the proof of Proposition A.1.

Lemma A.1 *Let $\kappa > 0$, $c > 0$, $\alpha > 0$ and $A \in [-\infty, +\infty)$ be constants. Let $b(x)$ be a twice differentiable function such that $b'(x)/x^{\alpha-1} \to 0$ and $b''(x)/x^{\alpha-2} \to 0$ as $x \to +\infty$. Suppose $p(x) = ce^{-\kappa|x|^\alpha + b(x)} I\{x > A\}$ is a probability density function. Then*

$$1 - F(x) = \frac{p(x)}{\kappa\alpha x^{\alpha-1}}\left(1 + O\left(\frac{1 + x|b'(x)|}{x^\alpha}\right)\right)$$

as $x \to +\infty$, where $F(x) = \int_A^x p(t)\, dt$ for $x > A$.

Proof of Lemma A.1. First, $1 - F(x) = \int_x^{+\infty} p(t)\, dt$ for $x > A$. Second, observe that $p'(x) = p(x) \cdot d(\log p(x))/dx$. So $p(x) = p'(x)\left(d(\log p(x))/dx\right)^{-1}$. It is trivial to verify that

$$\frac{d(\log p(t))}{dt} = -\kappa\alpha t^{\alpha-1} + b'(t) \quad \text{and} \quad \frac{d^2(\log p(t))}{dt^2} = -\kappa\alpha(\alpha - 1)t^{\alpha-2} + b''(t) \tag{3.1}$$

for $t > A$. Since $b'(x)/x^{\alpha-1} \to 0$ as $x \to +\infty$, for any $\epsilon > 0$, there exists $B > \max\{A, 0\}$, such that $-\epsilon x^{\alpha-1} \le b'(x) \le \epsilon x^{\alpha-1}$ for all $x \ge B$. Integrating the three terms over $[B, t]$, then dividing t^α, and letting $t \to +\infty$, and letting $\epsilon \to 0^+$, we obtain that $b(x)/x^\alpha \to 0$ as $x \to +\infty$. This implies $p(t) \cdot (d(\log p(t))/dt)^{-1} \to 0$ as $t \to +\infty$ by (3.1). By integration by parts, we have that

$$\begin{aligned} 1 - F(x) &= \int_x^\infty p'(t)\left(\frac{d\log p(t)}{dt}\right)^{-1} dt \\ &= -p(x) \cdot \left(\frac{d\log p(x)}{dx}\right)^{-1} + \int_x^\infty p(t)\left(\frac{d^2\log p(t)}{dt^2}\right)\left(\frac{d\log p(t)}{dt}\right)^{-2} dt. \end{aligned}$$

Recall $1 - F(x) = \int_x^{+\infty} p(t)\, dt$. We have that

$$\left|1 - F(x) + p(x) \cdot \left(\frac{d\log p(x)}{dx}\right)^{-1}\right| \le D(1 - F(x))$$

as $x > \max\{A, 0\}$, where

$$D := \sup_{t \ge x}\left|\left(\frac{d^2\log p(t)}{dt^2}\right)\left(\frac{d\log p(t)}{dt}\right)^{-2}\right| = O(x^{-\alpha})$$

as $x \to +\infty$, by (3.1), and conditions $b'(x) = o(x^{\alpha-1})$ and $b''(x) = o(x^{\alpha-2})$ as $x \to +\infty$. Combining the last two assertions, we obtain that

$$1 - F(x) = -(1 + O(x^{-\alpha}))^{-1} \cdot p(x) \cdot \left(\frac{d\log p(x)}{dx}\right)^{-1}$$

as $x \to +\infty$. Now $(d(\log p(x))/dx)^{-1} = -(\kappa\alpha)^{-1}x^{1-\alpha}(1 + O(b'(x)x^{1-\alpha}))$ as $x \to +\infty$. Thus, by condition $b'(x)/x^{\alpha-1} \to 0$ as $x \to +\infty$, we have that

$$1 - F(x) = \frac{p(x)}{\kappa\alpha x^{\alpha-1}}\left(1 + O\left(\frac{1 + x|b'(x)|}{x^\alpha}\right)\right)$$

as $x \to +\infty$. ■

Proof of Proposition A.1. We need to show that

$$\frac{t}{g(1-t)} \cdot \left(\log\frac{1}{t}\right)^{(\alpha-1)/\alpha} - \frac{1}{\alpha\kappa^{1/\alpha}} = O\left(\frac{\log(\log(1/t))}{\log(1/t)}\right) \tag{3.2}$$

as $t \to 0^+$ for case (i), and

$$\frac{t}{g(1-t)} \cdot \left(\log\frac{1}{t}\right)^{(\alpha-1)/\alpha} - \frac{1}{\alpha\kappa^{1/\alpha}} = O\left(\frac{1}{\log\log(1/t)}\right) \tag{3.3}$$

as $t \to 0^+$for case (ii), and if $b'(x) = o(x^{\alpha-1}/\log x)$ as $x \to +\infty$, then (3.3) still holds if the "O" on right hand side is replaced by "o".

First, by Lemma A.1,

$$\log(1 - F(x))^{-1} = \kappa x^\alpha - b(x) + O(\log x) \tag{3.4}$$

as $x \to +\infty$. Thus, by Lemma A.1 again, for both case (i) and case (ii),

$$\begin{aligned}
&\frac{1-F(x)}{p(x)}\left(\log\frac{1}{1-F(x)}\right)^{(\alpha-1)/\alpha} \\
= &\frac{1}{\alpha\kappa^{1/\alpha}}\left(1 + O\left(\frac{1 + x|b'(x)|}{x^\alpha}\right)\right)\left(1 + O\left(\frac{|b(x)| + \log x}{x^\alpha}\right)\right)^{(\alpha-1)/\alpha} \\
= &\frac{1}{\alpha\kappa^{1/\alpha}}\left(1 + O\left(\frac{\log x + |b(x)| + x|b'(x)|}{x^\alpha}\right)\right)
\end{aligned} \tag{3.5}$$

as $x \to +\infty$.

(i) Reviewing (3.2), let $x = F^{-1}(1-t)$, then $t = 1 - F(x)$. Therefore, (3.2) is equivalent to that

$$\frac{1-F(x)}{p(x)}\left(\log\frac{1}{1-F(x)}\right)^{(\alpha-1)/\alpha} - \frac{1}{\alpha\kappa^{1/\alpha}} = O\left(\frac{\log\log(1-F(x))^{-1}}{\log(1-F(x))^{-1}}\right)$$

as $x \to +\infty$. By (3.4), the right hand side above is $O((\log x)/x^\alpha)$ as $x \to +\infty$. Therefore the above follows by (3.5) and the assumption $b(x) = 0$.

(ii) Identity (3.3) is equivalent to that

$$\frac{1-F(x)}{p(x)}\left(\log\frac{1}{1-F(x)}\right)^{(\alpha-1)/\alpha} - \frac{1}{\alpha\kappa^{1/\alpha}} = O\left(\frac{1}{\log\log(1-F(x))^{-1}}\right) \tag{3.6}$$

as $x \to +\infty$. We first claim that condition $b'(x) = O(x^{\alpha-1}/\log x)$ implies that $b(x) = O(x^\alpha/\log x)$ as $x \to +\infty$. If this is true, the right hand side of (3.6) is equal to $O(1/\log x)$ from (3.4), and the left hand side is $O(1/\log x)$ by (3.5). This means that (3.6) holds. Now we prove our claim. Since

$b'(x) = O(x^{\alpha-1}/\log x)$ as $x \to +\infty$, there exists $x_0 > e$ and $K > 0$ such that $|b'(x)| \le Kx^{\alpha-1}/\log x$ as $x \ge x_0$. It follows that $|b(x)| \le |b(x_0)| + K\int_{x_0}^{x} t^{\alpha-1}(\log t)^{-1}\,dt$ for $x \ge x_0$. By L'Hospital's rule, it is easy to check that

$$\lim_{x\to+\infty} \frac{\int_{x_0}^{x} t^{\alpha-1}(\log t)^{-1}\,dt}{x^{\alpha}(\log x)^{-1}} = \frac{1}{\alpha}.$$

Therefore, $b(x) = O(x^{\alpha}/\log x)$ as $x \to +\infty$. Similarly, condition $b'(x) = o(x^{\alpha-1}/\log x)$ as $x \to +\infty$ implies that $b(x) = o(x^{\alpha}/\log x)$ as $x \to +\infty$. Then the second claim follows. ■

Proof of Proposition A.2. We will first prove that

$$(\log t^{-1})^{-1/\alpha} F^{-1}(1-t) \to \alpha\beta^{-1} \tag{3.7}$$

as $t \to 0^+$. Let $\epsilon \in (0,1)$. Note that $\left(F^{-1}(1-t)\right)' = -1/g(1-t)$ for $t \in (0,1)$. By the given condition, there exists $\delta \in (0,1)$ depending on ϵ such that

$$(1-\epsilon)\frac{1}{\beta t}\left(\log\frac{1}{t}\right)^{(1-\alpha)/\alpha} \le -\frac{dF^{-1}(1-t)}{dt} \le (1+\epsilon)\frac{1}{\beta t}\left(\log\frac{1}{t}\right)^{(1-\alpha)/\alpha}$$

for all $0 < t \le \delta$. Integrating the above over $[t,\delta]$, we obtain from the second inequality that

$$F^{-1}(1-t) - F^{-1}(1-\delta) \le (1+\epsilon)\int_t^{\delta} \frac{1}{\beta s}\left(\log\frac{1}{s}\right)^{(1-\alpha)/\alpha} ds$$

for any $0 < t \le \delta$. Now $((-\log s)^{1/\alpha})' = -(\alpha s)^{-1}(-\log s)^{(1-\alpha)/\alpha}$. Thus,

$$F^{-1}(1-t) \le F^{-1}(1-\delta) + (1+\epsilon)\alpha\beta^{-1}\left(\log\frac{1}{t}\right)^{1/\alpha} - (1+\epsilon)\alpha\beta^{-1}\left(\log\frac{1}{\delta}\right)^{1/\alpha}.$$

Letting $t \to 0^+$ first, then $\epsilon \to 0^+$, we obtain that

$$\limsup_{t\to 0^+} \frac{F^{-1}(1-t)}{(\log t^{-1})^{1/\alpha}} \le \frac{\alpha}{\beta}.$$

Similarly, we have $\liminf_{t\to 0^+}(\log t^{-1})^{-1/\alpha}F^{-1}(1-t) \ge \alpha\beta^{-1}$. Thus, (3.7) follows.

Given $\eta \in (0, \alpha\beta^{-1})$, by (3.7), there exists $b \in (0,1)$ such that

$$(\alpha\beta^{-1} - \eta)\left(\log\frac{1}{t}\right)^{1/\alpha} \le F^{-1}(1-t) \le (\alpha\beta^{-1} + \eta)\left(\log\frac{1}{t}\right)^{1/\alpha}$$

for any $t \in (0,b]$. The second inequality says that $1 - F((\alpha\beta^{-1}+\eta)\left(\log t^{-1}\right)^{1/\alpha}) \le t$. Let $x = (\alpha\beta^{-1}+\eta)\left(\log t^{-1}\right)^{1/\alpha}$. Then, $t = \exp(-x^{\alpha}/(\alpha\beta^{-1}+\eta)^{\alpha})$. Thus

$$1 - F(x) \le e^{-x^{\alpha}/(\alpha\beta^{-1}+\eta)^{\alpha}}$$

for $x \ge (\alpha\beta^{-1}+\eta)\left(\log b^{-1}\right)^{1/\alpha}$. Similarly, we get $1 - F(x) \ge e^{-x^{\alpha}/(\alpha\beta^{-1}-\eta)^{\alpha}}$ for $x \ge (\alpha\beta^{-1} - \eta)\left(\log b^{-1}\right)^{1/\alpha}$. The Proposition is proved because $\eta \in (0,\alpha\beta^{-1})$ is arbitrary. ■

Proposition A.3 *Let X satisfy the log-normal distribution, that is, $\log X \sim N(0,1)$. Then the conclusion of Theorem 1 does not hold. However, $(1/n)\sum_{i=1}^n \delta_{\tilde{D}_i}$ converges to the Gamma distribution with density $p(x) = 2^k x^{k-1} e^{-2x} I(x \ge 0)/(k-1)!$, where $\tilde{D}_i = \sqrt{i/2}(\log R_{k+i} - \log R_i)$ for $i \ge 1$.*

Proof. Since $X_i = e^{\log X_i}$ for $i \ge 1$, and $\{\log X_i;\ i \ge 1\}$ are i.i.d. $N(0,1)$, we have that

$$\mathcal{L}(R_1, R_2, \cdots, R_n) = \mathcal{L}\left(e^{\tilde{X}_{L_1}}, e^{\tilde{X}_{L_2}}, \cdots, e^{\tilde{X}_{L_n}}\right),\ n \ge 1,$$

where $\tilde{X}_1, \tilde{X}_2, \cdots$ are i.i.d. $N(0,1)$-random variables. Then $\tilde{D}_i = \sqrt{i/2}(\log R_{k+i} - \log R_i),\ 1 \le i \le n$, has the same distribution as that of $\sqrt{i}(2^{-1/2}\tilde{X}_{L_{k+i}} - 2^{-1/2}\tilde{X}_{L_i}),\ 1 \le i \le n$. Noticing $(1/\sqrt{2})N(0,1)$ has density $p(x) = \pi^{-1/2}e^{-x^2}$. By (i) of Proposition A.1 and Theorem 1, $\alpha = 2,\ \kappa = 1$, and

$$\frac{1}{n}\sum_{i=1}^n \delta_{\tilde{D}_i} \text{ converges weakly to } G_1 \ \ a.s. \tag{3.8}$$

with density $p(x) = 2^k x^{k-1} e^{-2x} I(x \ge 0)/(k-1)!$. If Theorem 1 holds, then there exists $\beta \in \mathbb{R}$ such that

$$\frac{1}{n}\sum_{i=1}^n I(i^\beta (R_{k+i} - R_i) \le x) \to G_2(x) \ a.s. \tag{3.9}$$

for any $x \in \mathbb{R}$ as $n \to \infty$, where $G_2(x)$ is the cumulative distribution function of a Gamma distribution. By the Mean-value theorem,

$$\log R_{k+i} - \log R_i \le \frac{R_{k+i} - R_i}{R_i}.$$

Therefore, recalling the definition of D_i,

$$\begin{aligned} A_n : &= \frac{1}{n}\sum_{i=1}^n I\left(i^\beta (R_{k+i} - R_i) \le x i^{\beta-(1/2)} R_i\right) \\ &\le \frac{1}{n}\sum_{i=1}^n I\left(\tilde{D}_i \le \frac{x}{\sqrt{2}}\right). \end{aligned} \tag{3.10}$$

We claim that

$$\liminf_{i \to +\infty} \frac{R_i}{e^{\sqrt{i}}} = +\infty \ \ a.s. \tag{3.11}$$

If this is true, by (3.8) and (3.9), there exists Ω' such that $P(\Omega') = 1,\ e^{-\sqrt{i}}R_i(\omega) \to +\infty$,

$$\frac{1}{n}\sum_{i=1}^n I\left(\tilde{D}_i(\omega) \le 2^{-1/2}x\right) \to G_1(2^{-1/2}x) \ \text{ and} \tag{3.12}$$

$$\frac{1}{n}\sum_{i=1}^n I(i^\beta (R_{k+i}(\omega) - R_i(\omega)) \le x) \to G_2(x) \tag{3.13}$$

as $n \to \infty$ for all $\omega \in \Omega'$ and all rational number x. Fix integer $m \ge 1,\ \omega \in \Omega'$, and rational number $x > 0$. Then there exists $i_0 \ge 1$ such that $x i^{\beta-(1/2)} R_i(\omega) \ge m$ for all $i \ge i_0$. Thus

$$A_n(\omega) = A_n \ge \frac{1}{n}\sum_{i=i_0}^n I\left(i^\beta (R_{k+i}(\omega) - R_i(\omega)) \le m\right) \to G_2(m)$$

by (3.13) as $n \to \infty$. Since m is arbitrary, $\lim_{n\to\infty} A_n(\omega) = 1$. However, by (3.10) and (3.12), $\limsup_{n\to\infty} A_n(\omega) \le G_1(2^{-1/2}x) < 1$. This yields a contradiction.

Now we prove (3.11). Note that R_i and $F^{-1}(1 - e^{-S_i})$ have the same distribution, where $F(x) = \Phi(\log x)$ for $x > 0$, and $S_i = \xi_1 + \cdots + \xi_i$, $i \ge 1$, and ξ_i's are i.i.d. *Exp*(1)-distributed random variables. Recalling that $1 - \Phi(x) \sim (1/\sqrt{2\pi}x)e^{-x^2/2}$ as $x \to +\infty$, we have that $1 - F(x) \sim (\sqrt{2\pi}\log x)^{-1} e^{-(\log x)^2/2}$ as $x \to +\infty$. Therefore, $-\log(1 - F(ie^{\sqrt{i}})) \sim i/2$ as $i \to +\infty$. It follows that

$$\begin{aligned} P\left(R_i \le ie^{\sqrt{i}}\right) &= P\left(S_i \le -\log(1 - F(ie^{\sqrt{i}}))\right) \\ &\le P\left(\frac{S_i}{i} \le \frac{2}{3}\right) \\ &\le e^{-Ki} \end{aligned} \tag{3.14}$$

for i large enough, where $K > 0$ is a constant resulted in using (i) of Lemma A.2 below in the last step; we also use the fact that $E\xi_1 = 1$ and that $I(x)$ is positive, non-increasing for $x \in (0,1)$ in that step. Thus, $\sum_{i\ge 1} P\left(R_i \le ie^{\sqrt{i}}\right) < \infty$. This leads to the desired conclusion by the Borel-Cantelli lemma. ■

The first part of next lemma is (c) of Remarks on page 27 from [10]; The second part corresponds to Theorem 3.7.1 on page 109 from [10] when $d = 1$ and $\mathbf{C} = \sigma^2$.

Lemma A.2 *Let $\{\xi, \xi_i, i = 1, 2, \cdots\}$ be a sequence of i.i.d. random variables. Let $S_n = \sum_{i=1}^n \xi_i, n \ge 1$. Then*

(i) For any $A \subset \mathbb{R}$ and $n \ge 1$,

$$P(S_n/n \in A) \le 2e^{-nI(A)},$$

where $I(x) = \sup_{t\in\mathbb{R}}\{tx - \log E(e^{t\xi})\}$ and $I(A) = \inf_{x\in A} I(x)$.

(ii) Assume further that $E\xi = 0$, $Var(X) = \sigma^2 > 0$ and $Ee^{t_0\xi} < \infty$ for some $t_0 > 0$. Let $\{a_n; n = 1, 2, \cdots\}$ be a sequence of positive numbers such that $a_n \to 0$ and $na_n \to \infty$ as $n \to \infty$. Then

$$\lim_{n\to\infty} a_n \log P\left(\sqrt{\frac{a_n}{n}} S_n \in A\right) = -\inf_{x\in A}\left\{\frac{x^2}{2\sigma^2}\right\}$$

for any subset $A \subset \mathbb{R}$ such that $\inf\{|x|; x \in A^\circ\} = \inf\{|x|; x \in \bar{A}\}$.

The following Poisson approximation theorem is a special case of Theorem 1 from [2], which is again a special case of the Stein Poisson approximation method, see [3], [24], [25] and literatures therein. One application of the theorem is studying behaviors of maxima of random variables; see, for example, [17], [18] and [19].

Lemma A.3 *Let I be an index set and $\{B_\alpha, \alpha \in I\}$ be a set of subsets of I, that is, $B_\alpha \subset I$ for each $\alpha \in I$. Let $\{\eta_\alpha, \alpha \in I\}$ be random variables. For a given $t \in \mathbb{R}$, set $\lambda = \sum_{\alpha\in I} P(\eta_\alpha > t)$. Then*

$$|P(\max_{\alpha\in I} \eta_\alpha \le t) - e^{-\lambda}| \le (1 \wedge \lambda^{-1})(b_1 + b_2 + b_3)$$

where

$$
\begin{aligned}
b_1 &= \sum_{\alpha\in I}\sum_{\beta\in B_\alpha} P(\eta_\alpha > t)P(\eta_\beta > t),\\
b_2 &= \sum_{\alpha\in I}\sum_{\alpha\neq\beta\in B_\alpha} P(\eta_\alpha > t, \eta_\beta > t),\\
b_3 &= \sum_{\alpha\in I} E|P(\eta_\alpha > t|\sigma(\eta_\beta, \beta\notin B_\alpha)) - P(\eta_\alpha > t)|,
\end{aligned}
$$

and $\sigma(\eta_\beta, \beta\notin B_\alpha)$ is the σ-algebra generated by $\{\eta_\beta, \beta\notin B_\alpha\}$. Particularly, if η_α is independent of $\{\eta_\beta, \beta\notin B_\alpha\}$ for each α, then $b_3 = 0$.

Lemma A.4 *Let $\xi, \xi_1, \xi_2, \cdots$ be i.i.d. Exp(1)-distributed random variables with $S_j = \xi_1 + \xi_2 + \cdots + \xi_j$, $j\geq 1$. Given integer $k\geq 1$, define*

$$W_n = \max\{S_{k+1} - S_1, S_{k+2} - S_2, \cdots, S_{k+n} - S_n\}$$

for $n\geq 1$. Then

$$P(W_n \leq \log n + (k-1)\log(\log n) + x) \to \exp\left(-\frac{e^{-x}}{(k-1)!}\right)$$

for any $x\in\mathbb{R}$ as $n\to+\infty$.

Proof. Set $I = \{1, 2, \cdots, n\}$ and

$$
\begin{aligned}
&B_i = \{\beta = \{j+1, j+2, \cdots, j+k\};\ \beta\cap\{i+1, i+2, \cdots, i+k\}\neq\emptyset\};\\
&\eta_i = S_{k+i} - S_i \ \text{ and } \ t = t_n = \log n + (k-1)\log(\log n)) + x.
\end{aligned}
$$

By L'Hospital's rule, one can easily check that

$$P(S_k > t) = \int_t^\infty \frac{x^{k-1}e^{-x}}{(k-1)!}\,dx \sim \frac{t^{k-1}e^{-t}}{(k-1)!} \tag{3.15}$$

as $t\to+\infty$. Then

$$\lambda = \sum_{i=1}^n P(S_{k+i} - S_i > t) = nP(S_k > t) \sim n\frac{t^{k-1}e^{-t}}{(k-1)!} \sim \frac{e^{-x}}{(k-1)!} \tag{3.16}$$

as $n\to\infty$. Recall Lemma A.3, by independence, $b_3 = 0$. Note that $\#B_i \leq 3k$. We have from (3.15) that

$$b_1 \leq 3knP(S_k > t)^2 = O\left(\frac{k}{n}\right)$$

as $n\to\infty$. Moreover,

$$
\begin{aligned}
b_2 &\leq 3kn \max_{2\leq i\leq k+1} P(S_{k+i} - S_i > t, S_{k+1} - S_1 > t)\\
&\leq 3knP(S_{k+2} - S_2 > t, S_{k+1} - S_1 > t)\\
&= 3knP(S_{k+1} - S_1 > t, S_k > t)
\end{aligned}
$$

where the second inequality is obtained by Lemma 2.1 from [18]. Now we estimate the last probability. First, by independence,

$$\begin{aligned} P(S_{k+1} - S_1 > t, S_k > t) &= E\left(P(S_k > t|\xi_2, \cdots, \xi_k)^2\right) \\ &= E\left(P(S_k > t|\xi_2, \cdots, \xi_k)^2 (I(A) + I(A^c))\right) \end{aligned}$$

where $A = \{S_k - S_1 > t\}$. If $k = 1$, $P(A) = 0$. Otherwise, $S_k - S_1 \sim Gamma(k-1)$. It then follows from (3.15) that the above is identical to

$$\begin{aligned} P(A) + Ee^{-2(t-S_k+S_1)} I(S_k - S_1 \le t) &\le \frac{t^{k-2}e^{-t}}{(k-2)!} + e^{-2t}\int_0^t \frac{y^{k-2}e^{-y}}{(k-2)!} e^{2y}\, dy \\ &\le t^{k-2}e^{-t} + e^{-2t}\int_0^t y^{k-2}e^y\, dy. \end{aligned}$$

By L'Hospital's rule,

$$\int_0^t y^{k-2}e^y\, dy \sim t^{k-2}e^t$$

as $t \to +\infty$. In summary, we have from (3.16) that

$$b_2 \le 9knt^{k-2}e^{-t} = O\left(\frac{1}{t}\right) = O\left(\frac{1}{\log n}\right)$$

as $n \to \infty$. Consequently, $b_i \to 0$ as $n \to \infty$ for $i = 1, 2, 3$. The lemma is concluded by Lemma A.3.
■

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