THE UNIVERSITY OF MINNESOTA

Statistics 5401 September 25, 2005

Drawing ellipses in MacAnova

There are several ways to draw ellipses in MacAnova.

The defining equation for an ellipse centered at $\mathbf{x}_0 = [x_{10}, x_{20}]'$ and with shape matrix $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$ is

$$q^{11}(x_1 - x_{10})^2 + 2q^{12}(x_1 - x_{10})(x_2 - x_{20}) + q^{22}(x_2 - x_{20})^2 = K^2$$

where q^{jk} are the elements of $\mathbf{Q}^{-1}$.

When you solve for x_2 in terms of x_1, you get the following equation:

$$x_2 = x_{20} - q^{12}(x_1 - x_{10})/q^{22} \pm \{K^2/q^{22} - (q^{11}q^{22} - (q^{12})^2)(x_1 - x_{10})^2/(q^{22})^2\}^{1/2}.$$

The + and – signs go with the top and bottom halves of the ellipse, respectively.

This works only when the expression in {...} is nonnegative. This us the case only when $x_{10} - K\sqrt{q_{11}} \le x_1 \le x_{10} + K\sqrt{q_{11}}$, that is $|x_1 - x_{10}| \le K\sqrt{q_{11}}$. When $|x_1 - x_{10}| > K\sqrt{q_{11}}$, no real number x_2 satisfies the equation.

Similarly, you can express x_1 in terms of x_2 by

$$x_1 = x_{10} - q^{12}(x_2 - x_{20})/q^{11} \pm \{K^2/q^{11} - (q^{11}q^{22} - (q^{12})^2/(q^{11})^2\}^{1/2}$$

For this to be meaningful, x_2 must satisfy $x_{20} - K\sqrt{q_{22}} \le x_2 \le x_{20} + K\sqrt{q_{22}}$, that is $|x_2 - x_{20}| \le K\sqrt{q_{22}}$.

Here I apply this in MacAnova:

```
Cmd> Q <- matrix(vector(25,10.5,10.5,9),2); Q
(1,1)          25        10.5 Positive definite symmetric matrix
(2,1)        10.5           9

Cmd> x0 <- vector(30,40) # center

Cmd> K <- sqrt(invchi(.95, 2)); K # constant defining size
(1)      2.4477

Cmd> Qinv <- solve(Q)

Cmd> x1min <- x0[1] - K*sqrt(Q[1,1]) # minimum possible value for x1

Cmd> x1max <- x0[1] + K*sqrt(Q[1,1]) # maximum possible value for x1
```

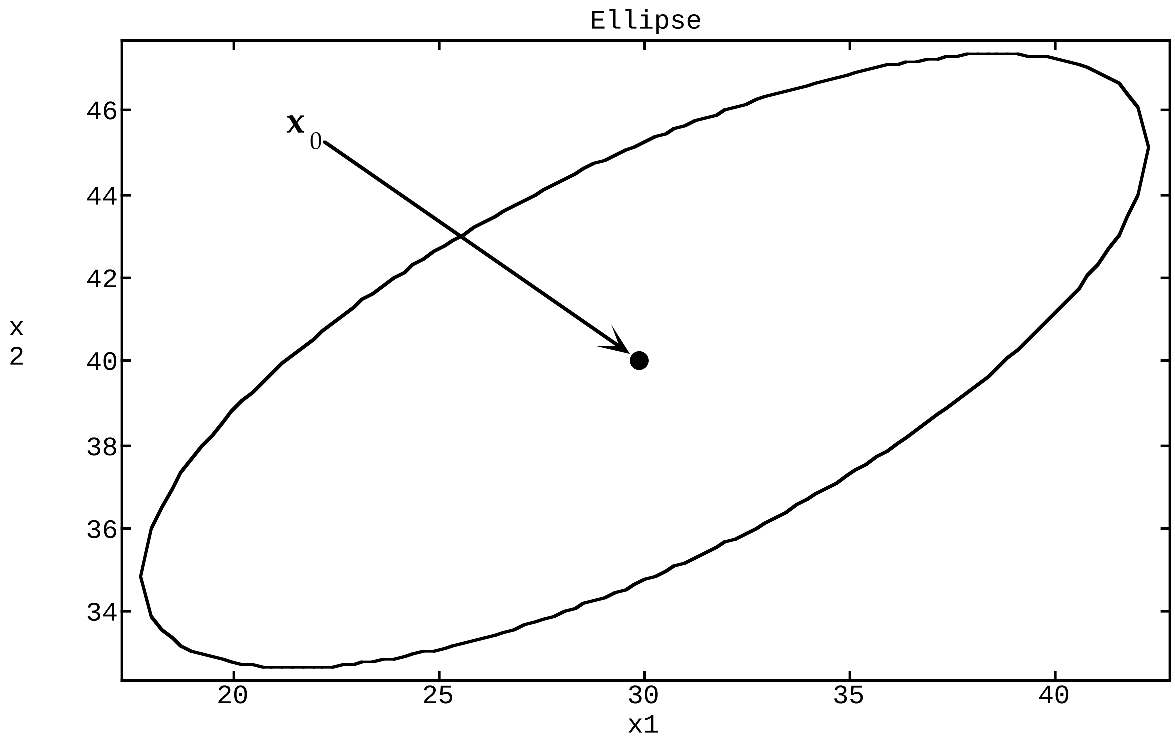
```
Cmd> x1 <- x1min + (x1max - x1min)*run(0,100)/100 #values for x1
```

`x1` now contains 101 equally spaced values from `x1min` to `x1max`. Now I computed x_2 values for the top and bottom of the ellipse.

```
Cmd> x2top <- x0[2] - Qinv[1,2]*(x1 - x0[1])/Qinv[2,2] + \
    sqrt(K^2/Qinv[2,2] -\
    (Qinv[1,1]*Qinv[2,2] - Qinv[1,2]^2)*(x1 - x0[1])^2/Qinv[2,2]^2)
Cmd> x2bottom <- x0[2] - Qinv[1,2]*(x1 - x0[1])/Qinv[2,2] - \
    sqrt(K^2/Qinv[2,2] -\
    (Qinv[1,1]*Qinv[2,2] - Qinv[1,2]^2)*(x1 - x0[1])^2/Qinv[2,2]^2)
```

Vectors `x2top` and `x2bottom` now contain the x_2 coordinates for the top and bottom halves of the ellipse.

```
Cmd> lineplot(x1,x2top,show:F) # draw but don't show top half
Cmd> addlines(x1,x2bottom,ymin:?,title:"Ellipse",xlab:"x1",ylab:"x2")
```



' Its much easier to do this using macro `ellipse`.

```
Cmd> help(ellipse:vector("usage","plotting_ellipse"))
Subtopic 'usage' of help on 'ellipse'
You can use ellipse() to compute and optionally draw an ellipse with
shape defined by a specified positive definite matrix and centered at
a specified point

ellipse(K, Q [,x0] [,graphics keywords]) computes xvals and yvals, the
x- and y-coordinates of points on the ellipse defined by the equation
        (x - x0)' %*% solve(Q) %*% (x - x0) = K^2
The value returned is structure(x:xvals,y:yvals [,graphics keywords]).

K > 0 must be a REAL scalar and Q must be a 2 by 2 REAL positive
definite symmetric matrix.  If x0 is an argument, it must be a REAL
vector of length 2.  Otherwise, rep(0,2) is used for x0.

Subtopic 'plotting_ellipse' of help on 'ellipse'
The ellipse can be plotted by
  Cmd> result <- ellipse(K, Q [,x0] [,graphics keywords])

  Cmd> lineplot(keys:result)

ellipse(K, Q [,x0], draw:T [,graphics keywords]) draws the ellipse
directly and doesn't return the coordinates as a value.  If the
ellipse is to be added to an existing graph, include add:T as an
argument.
```

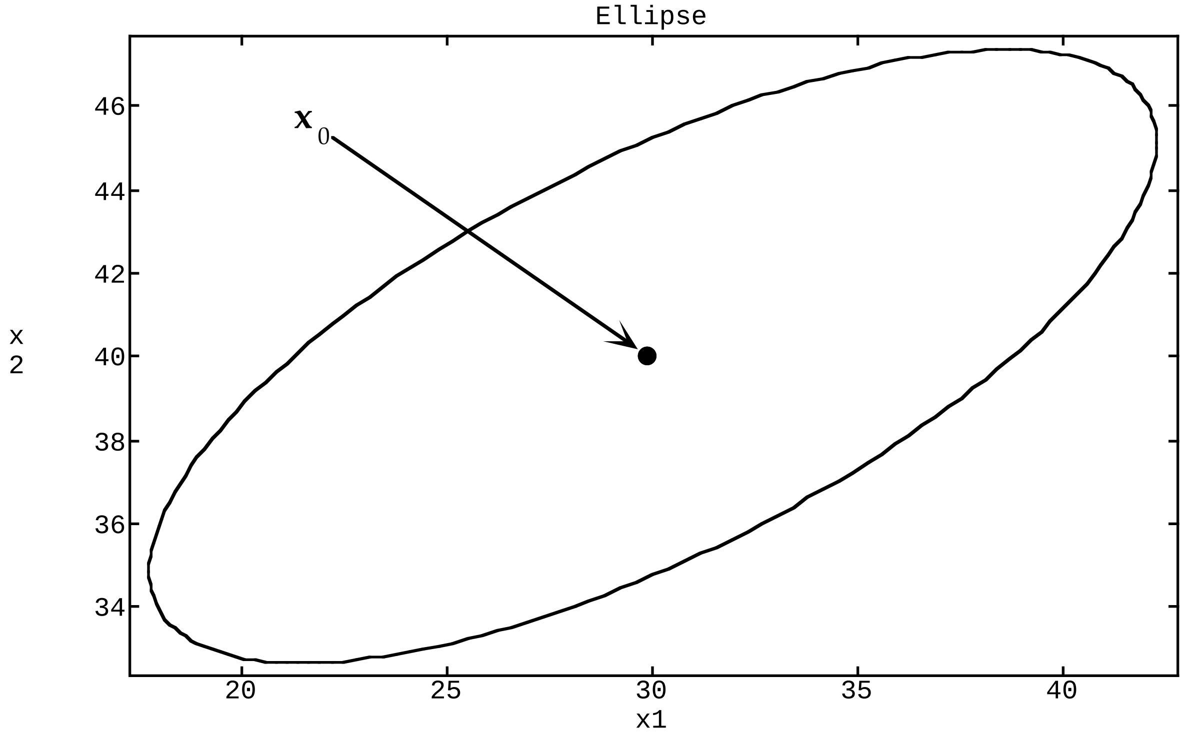
So lets use `ellipse()`:

```
Cmd> coords <- ellipse(K,Q,x0) # compute coordinates for ellipse
```

`coords` is a structure with two components, `x` and `y`:

```
Cmd> compnames(coords) # coords includes both x and y
(1) "x"
(2) "y"
```

```
Cmd> lineplot(coords$x,coords$y,title:"Ellipse",xlab:"x1",ylab:"x2")
```



You can actually just use `coords` itself instead of its `x` and `y` components separately. The following produces the identical plot.

```
Cmd> lineplot(coords,title:"Ellipse",xlab:"x1",ylab:"x2")
```

With keyword phrase `draw:T` and other graphics keyword phrases, `ellipse()` will draw the ellipse itself. The following single command will reproduce the preceding plot:

```
Cmd> ellipse(K,Q,x0,title:"Ellipse",xlab:"x1",ylab:"x2",draw:T)
```

By including `add:T`, you can add the ellipse to an existing plot. In the following lines, I generated and made a scatter plot of a bivariate normal sample with $\boldsymbol{\mu} = \mathbf{x}_0$ and $\boldsymbol{\Sigma} = \mathbf{Q}$, and then drew the following contour of the normal distribution:

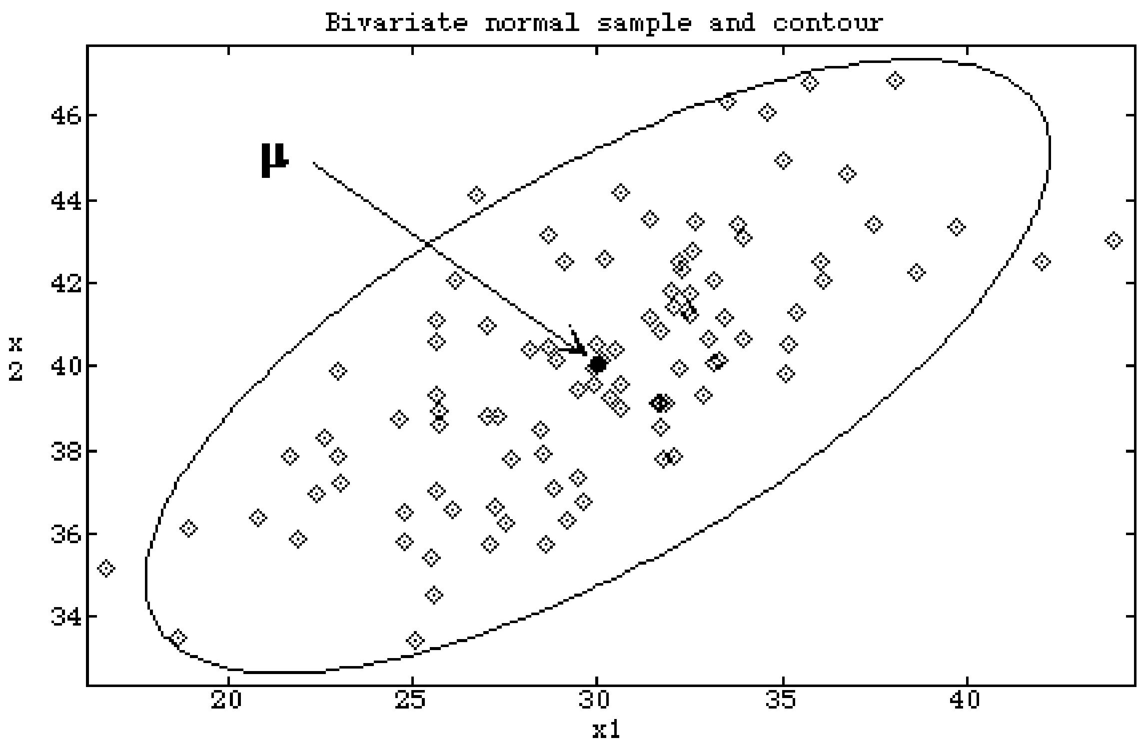
$$\sigma^{11}(x_1 - \mu_1)^2 + 2\sigma^{12}(x_1 - \mu_1)(x_2 - \mu_2) + \sigma^{22}(x_2 - \mu_2)^2 = K^2$$

where σ^{ij} are elements of $\boldsymbol{\Sigma}^{-1}$, and $K^2 = \chi_2^2(.95) = 5.991$ is a probability point of χ^2 on 2 degrees of freedom. With this value of K, the contour encloses 95% of the population.

```
Cmd> sigma <- Q; mu <- x0

Cmd> n <- 100; y <- rmvnorm(n, sigma, mu) # N_2(mu,sigma) sample
```

`rmvnorm(n, sigma, mu)` computes a multivariate normal sample of size `n`, variance matrix `sigma` and mean vector `mu`. Look at the help for full details.

```
Cmd> covar(y) # sample mean vector and variance matrix
WARNING: searching for unrecognized macro covar near covar(
component: n
(1)           100
component: mean
(1,1)      29.888      39.957  Pretty close to μ
component: covariance          Pretty close to Σ
(1,1)      24.066       9.974
(2,1)       9.974      8.4978

Cmd> plot(y[,1],y[,2],symbol:"\1",title:"Bivariate normal sample",\
xlab:"y1",ylab:"y2",show:F) # make scatter plot but don't show it

Cmd> ellipse(K,sigma,mu,draw:T,add:T,xmin:?,xmax:?,ymin:?,ymax:?,\
title:"Bivariate normal sample and contour",xlab:"x1",ylab:"x2")
```



Keyword phrases `xmin:?`,`xmax:?`,`ymin:?` and `ymax:?` ensure the limits of the graph includes all the points and the contour. Without them, you might find some of the the contour was cut off by the frame.