Displays for Statistics 5401/8401

Lecture 18

October 17, 2005

Christopher Bingham, Instructor

612-625-1024, kb@umn.edu
372 Ford Hall

Class Web Page

http://www.stat.umn.edu/~kb/classes/5401

Confidence Intervals Continued

Model is $y_{ij} = \mu + \alpha_j + \varepsilon_{ij}$.

```
Cmd> manova("y=varieties",silent:T)
Cmd> stats <- secoefs()#info on last regress(),anova(),manova()
Cmd> stats
component: CONSTANT        Estimates and their standard errors
  component: coefs         Least squares estimates of μ
          SepLen      SepWid      PetLen      PetWid
(1)       5.8433      3.0573       3.758      1.1993  μ̂'

  component: se            Their standard errors
          SepLen      SepWid      PetLen      PetWid
(1)     0.042032    0.027735    0.035137     0.01671
component: varieties
  component: coefs         Least squares of variety effects
          SepLen      SepWid      PetLen      PetWid
(1)     -0.83733     0.37067      -2.296    -0.95333  α̂'1
(2)     0.092667    -0.28733       0.502     0.12667  α̂'2
(3)      0.74467   -0.083333       1.794     0.82667  α̂'3

  component: se         Their standard errors
          SepLen      SepWid      PetLen      PetWid
(1)     0.059443    0.039224    0.049691    0.023631
(2)     0.059443    0.039224    0.049691    0.023631
(3)     0.059443    0.039224    0.049691    0.023631
Cmd> alphahat1 <- vector(stats$varieties$coefs[1,]); alphahat1
(1)     -0.83733     0.37067      -2.296    -0.95333
Cmd> ses <- vector(stats$varieties$se[1,]); ses # std errors
(1)     0.059443    0.039224    0.049691    0.023631
```

- `stats$varieties$coefs[1,]` gets the first row $\hat{\alpha}_1$ of the matrix of estimated variety effect coefficients.
- `stats$varieties$se[1,]` gets their standard errors.

I will calculate several types of 99% confidence limits for the p = 4 elements $\alpha_{\ell 1}$ of α_1.

```
Cmd> n <- nrows(y); p <- ncols(y)
Cmd> g <- 3 # number of groups
Cmd> fe <- reverse(DF)[1] # or DF[3]  or  n - g
Cmd> vector(n, p, fe, g)
(1)         150           4         147           3
```

All confidence intervals are of the form $\hat{\alpha}_{1\ell} \pm K\times \widehat{SE}[\hat{\alpha}_{1\ell}]$, ($\widehat{SE}$ means estimated SE)

Individual (non simultaneous) confidence limits

Use ordinary Student's t, $K = t_{f_e}(\alpha/2)$

```
Cmd> alpha <- .01 # .99 = 1 - alpha
Cmd> tcrit1 <- invstu(alpha/2, fe, upper:T); tcrit1
(1)      2.6097     non-bonferronized critical value
Cmd> alphahat1 + vector(-1,1)'*tcrit1*ses
(1,1)    -0.99246    -0.68221
(2,1)      0.2683     0.47303
(3,1)     -2.4257     -2.1663
(4,1)      -1.015    -0.89166
            Lower      Upper   limits
```

`vector(-1,1)'` codes for ±1.

The transpose is needed so the result comes out in 2 columns.

Simultaneous limits for α_{11}, α_{12}, α_{13}, α_{14} (elements of α_1), ignoring α_2 and α_3.

Bonferronize by p = 4: $K = t_{f_e}((\alpha/4)/2)$

```
Cmd> tcrit2 <- invstu((alpha/p)/2,fe,upper:T); tcrit2
(1)      3.0763
Cmd> alphahat1 + vector(-1,1)'*tcrit2*ses
(1,1)     -1.0202    -0.65447
(2,1)        0.25     0.49133
(3,1)     -2.4489     -2.1431
(4,1)      -1.026    -0.88064
```

These limits are 18% wider than non-Bonferronized limits (3.076 > 2.610).

Simultaneous limits for all 12 = g×p effects

Bonferronize by gp = 12: $K = t_{f_e}((\alpha/12)/2)$

```
Cmd> tcrit3 <- invstu((alpha/(g*p))/2,DF[3],upper:T); tcrit3
(1)      3.4119
Cmd> alphahat1 + vector(-1,1)'*tcrit3*ses
(1,1)     -1.0401    -0.63452
(2,1)     0.23684      0.5045
(3,1)     -2.4655     -2.1265
(4,1)      -1.034    -0.87271
```

These limits are wider still, 31% larger than non-simultaneous limits and 11% wider than the Bonferronized by 4 limits.

"Ellipsoidal" Limits simultaneous for α_{11}, α_{12}, α_{13}, α_{14} (elements of $\boldsymbol{\alpha}_1$): K = $\sqrt{T^2(\alpha)}$, $T^2(\alpha)$ a critical value for T^2.

```
Cmd> fe1 <- fe-p+1; tcrit4 <-\
        sqrt((p*fe/fe1)*invF(alpha,p,fe1,upper:T)); tcrit4
(1)      3.7545
Cmd> alphahat1 + vector(-1,1)'*tcrit4*ses
(1,1)     -1.0605    -0.61415
(2,1)      0.2234     0.51793
(3,1)     -2.4826     -2.1094
(4,1)     -1.0421    -0.86461
```

These are simultaneous for <u>all possible linear combinations</u> of $\hat{\alpha}_{11}$, $\hat{\alpha}_{12}$, $\hat{\alpha}_{13}$, and $\hat{\alpha}_{14}$. They are 22% wider than Bonferronized by 4 limits.

How do you extend this approach to all 12 $\alpha_{j\ell}$?

One way is to Bonferronize these limits. by g = 3: K = $T^2(\alpha/3)$

```
Cmd> tcrit5 <- sqrt((p*fe/fe1)*invF(alpha/g,p,fe1,upper:T));\
     tcrit5
(1)      4.1108
Cmd> alphahat1 + vector(-1,1)'*tcrit5*ses
(1,1)     -1.0817    -0.59297
(2,1)     0.20942     0.53191
(3,1)     -2.5003     -2.0917
(4,1)     -1.0505    -0.85619
```


Testing Multivariate Linear Hypotheses

The k+1 by p matrix **B** of coefficients has columns $\mathbf{b}_\ell$ and rows $\boldsymbol{\beta}_j'$:

$$\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p] = [\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_k]'$$

Some <u>linear hypotheses</u> are:

- H_0: $\boldsymbol{\beta}_j = \mathbf{0}$ (y_ℓ does not depend on $\mathbf{Z}_j$ for $\ell = 1, 2, \ldots, p$)
  You can express this as
  H_0: $\boldsymbol{\ell}'\mathbf{B} = 0$, $\boldsymbol{\ell}' = \begin{array}{c} [0 \ldots 0 \;\; 1 \;\; 0 \ldots 0] \\ {\scriptstyle 0 \quad\; j-1 \;\; j \;\; j+1 \quad k} \end{array}$
- H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ (equal coefficients of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ for <u>all p variables</u>)
  You can express this as
  H_0: $\boldsymbol{\ell}'\mathbf{B} = 0$, $\boldsymbol{\ell}' = [0 \;\; 1 \;\; -1 \;\; 0 \ldots 0]$
- H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_k = 0$ (no effect of $Z_1, \ldots, Z_k$ on <u>any variable</u>.

- General H_0: $\mathbf{LB} = \mathbf{0}$, $\mathbf{L} = [\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_r]'$,

$$\mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1' \\ \boldsymbol{\ell}_2' \\ \vdots \\ \boldsymbol{\ell}_r' \end{bmatrix}$$

Each row $\boldsymbol{\ell}_i'$ of **L** defines a <u>linear combination</u>

$$\boldsymbol{\ell}_i'\mathbf{B} = \textstyle\sum_{0\le j\le k} \ell_{ij}\boldsymbol{\beta}_j'$$

of the *rows* $\boldsymbol{\beta}_j'$ of **B**. Also

$$\boldsymbol{\ell}_i'\mathbf{B} = [\boldsymbol{\ell}_i'\mathbf{b}_1 \;\; \boldsymbol{\ell}_i'\mathbf{b}_2 \;\; \ldots \;\; \boldsymbol{\ell}_i'\mathbf{b}_p]$$

$$\boldsymbol{\ell}_i'\mathbf{b}_\ell = \textstyle\sum_{0\le j\le k} \ell_{ij}\beta_{j\ell}$$

where $\mathbf{b}_\ell = [\beta_{0\ell}, \beta_{1\ell}, \ldots, \beta_{k\ell}]'$ is the vector of coefficients for y_ℓ.

The linear combination of coefficients *is the same for every variable*.

H_0 declares that r×p linear combinations are 0.

The alternative hypothesis considered is

$$H_1\text{:}\mathbf{LB} \ne \mathbf{0}$$

H_1 is true if at least one of the r×p linear combinations in **LB** is not zero.

Here's what **L** is for the examples

- H_0: $\boldsymbol{\beta}_j = \mathbf{0}$
  H_1: $\beta_{j\ell} \ne 0$ for at least one ℓ
  r = 1 and $\mathbf{L} = \begin{array}{c} [0 \;\; 0 \;\ldots\; 0 \;\; 1 \;\; 0 \ldots 0] \\ {\scriptstyle 0 \;\; 1 \qquad j-1 \;\; j \;\; j+1 \quad k} \end{array}$
- H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$
  H_1: $\beta_{1\ell} \ne \beta_{2\ell}$ for at least one ℓ
  r = 1 and $\mathbf{L} = [0 \;\; 1 \;\; -1 \;\; 0 \;\; 0 \ldots \;\; 0]$
- H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_k$
  H_1: $\beta_{j\ell} \ne 0$ for at least one j and ℓ

$$r = k,\; \mathbf{L} = [\mathbf{0}\;\; \mathbf{I}_k] = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot\,\cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

Because the same **L** applies to every variable, this formulation does *not* include some hypotheses you might think of as "linear."

Example:

H_0: $\beta_{12} = 0$

(variable 2 doesn't depend on Z_1)

You can't express this as **LB** = 0 for any **L** and can't test is by the methods I am about to discuss.

These methods do allow testing

H_0: $\beta_{11} = \beta_{12} = \ldots = \beta_{1p} = 0$

(*no* variable depends on Z_1).

Consider null and alternative linear hypotheses H_0: $\mathbf{LB} = 0$ and H_1: $\mathbf{LB} \neq 0$.

Suppose

- $\hat{\mathbf{B}}^0$ estimates **B** assuming H_0 is true, that is, by least squares, restricted so that $\mathbf{L}\hat{\mathbf{B}}^0 = \mathbf{0}$
- $\hat{\mathbf{B}}^1$ estimates **B** without assuming H_0 is true so $\mathbf{L}\hat{\mathbf{B}}^1 \neq 0$.

Define matrices of <u>sums of squares and products</u> of residuals

$$RCP(H_0) = \sum_{1\leq i\leq N}(\mathbf{y}_i - \hat{\mathbf{y}}_i^0)(\mathbf{y}_i - \hat{\mathbf{y}}_i^0)'$$
$$RCP(H_1) = \sum_{1\leq i\leq N}(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)'$$

where fitted values $\hat{\mathbf{y}}_i^0$ and $\hat{\mathbf{y}}_i^1$ are computed using $\hat{\mathbf{B}}^0$ and $\hat{\mathbf{B}}^1$. That is

$$[\hat{\mathbf{y}}_1^0, \hat{\mathbf{y}}_2^0, \ldots\ldots, \hat{\mathbf{y}}_N^0]' = \hat{\mathbf{Y}}^0 = \mathbf{Z}\hat{\mathbf{B}}^0 = \sum_j \mathbf{Z}_j(\hat{\boldsymbol{\beta}}_j^0)'$$
$$[\hat{\mathbf{y}}_1^1, \hat{\mathbf{y}}_2^1, \ldots\ldots, \hat{\mathbf{y}}_N^1]' = \hat{\mathbf{Y}}^1 = \mathbf{Z}\hat{\mathbf{B}}^1 = \sum_j \mathbf{Z}_j(\hat{\boldsymbol{\beta}}_j^1)'$$

The ***hypothesis matrix*** for H_0 is

$$\mathbf{H} \equiv RCP(H_0) - RCP(H_1)$$

- the *reduction* of $RCP(H_0)$ achieved by <u>not imposing</u> restrictions of H_0
- or the *increase* in $RCP(H_1)$ resulting from <u>imposing</u> those restrictions.

The ***error matrix*** is

$$\mathbf{E} = RCP(H_1) = \sum(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)'$$

In the one-way MANOVA case, **H** = **B** in and **E** = **W** J&W's notation.

- **H** is always <u>positive semi-definite</u> (all eigenvalues $\geq$ 0).
- When $\boldsymbol{\Sigma}$ is non-singular and the error d.f. = $f_e > p-1$ ($f_e-p+1 > 0$), **E** is positive definite (all eigenvalues > 0).
- When $f_e \leq p-1$ ($f_e-p+1 \leq 0$) **E** is not invertible but is positive semi-definite

A matrix principle of reduction in residual sums of squares and products

The "larger" **H** is compared to **E**, the better H_1 fits the data than H_0.

The testing principle is:

Reject H_0 in favor of H_1 when **H** is "large" as compared to **E**

This idea underlies all the tests we will consider: Wilks' (likelihood ratio), Hotelling's generalized T^2, Pillai's trace and Roy's maximum eigenvalue.

They are bassed on different answers to the **important question**

$\boxed{\text{How do you compare } \mathbf{H} \text{ with } \mathbf{E}?}$

Q <u>How do you compare **H** with **E**</u>?

There is no single good way to compare **H** with **E**.

Things are simplest when $p = 1$ or $f_h = 1$.

- When $p = 1$. This is the univariate case and you can choose between an F-test and Bonferronized t-tests.
- When $f_h = 1$, this is essentialy the case of a hypothesis about single vector of parameters $\boldsymbol{\gamma}$ such as $\boldsymbol{\gamma} = \boldsymbol{\mu}$ (1 sample) or $\boldsymbol{\gamma} = \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ (2 sample).

  Your choice is between a test based on $T^2 = \hat{\boldsymbol{\gamma}}'\hat{V}[\hat{\boldsymbol{\gamma}}]^{-1}\hat{\boldsymbol{\gamma}}$, $\hat{\boldsymbol{\gamma}}' = \mathbf{L}\hat{\mathbf{B}}$ and Bonferronized $t_\ell = \hat{\gamma}_\ell/\widehat{SE}[\hat{\gamma}_\ell]$. $1 \leq \ell \leq p$.

Things are more complicated when $p > 1$ and $f_h > 1$.

Summarize

- The <u>hypothesis matrix</u>

  $$\mathbf{H} \equiv RCP(H_0) - RCP(H_1)$$

  is a difference of matrices of sums of squares and products of residuals when H_0 and H_1 are fitted.
- The error matrix

  $$\mathbf{E} = RCP(H_1) = \sum(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)(\mathbf{y}_i - \hat{\mathbf{y}}_i^1)'$$

  is the matrix of sums of squares and products of residuals when H_1 is fitted.
- We reject H_0 when **H** is "large" when compared to **E**.

One-way MANOVA

The linear model is

$$\mathbf{y}_{ij} = \boldsymbol{\mu} + \boldsymbol{\alpha}_j + \boldsymbol{\varepsilon}_{ij},\ j = 1,\ldots,g,\ i = 1,\ldots,n_j$$

$$\sum_{1\leq j\leq g}\boldsymbol{\alpha}_j = \mathbf{0}.$$

- H_0: $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 = \ldots = \boldsymbol{\alpha}_g = 0$

  H_1: $\boldsymbol{\alpha}_{j_1} \neq \boldsymbol{\alpha}_{j_2}$, some $j_1 \neq j_2$
- $f_h = g - 1$ (same as univariate)
- $f_e = N - g$ (same as univariate)
- $\mathbf{H} = RCP(H_0) - RCP(H_1)$

  $= \sum_j\sum_i(\mathbf{y}_{ij}-\overline{\mathbf{y}}_{..})(\mathbf{y}_{ij}-\overline{\mathbf{y}}_{..})'-\sum_j\sum_i(\mathbf{y}_{ij}-\overline{\mathbf{y}}_{.j})(\mathbf{y}_{ij}-\overline{\mathbf{y}}_{.j})'$

  $= \sum_j n_j(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})(\overline{\mathbf{y}}_{.j} - \overline{\mathbf{y}}_{..})' = \mathbf{B}$ (J&W),

  where

  $\overline{\mathbf{y}}_{.j} = (1/n_j)\sum_{1\leq i\leq n_j}\mathbf{y}_{ij}$ = group j mean

  $\overline{\mathbf{y}}_{..} = (1/N)\sum_j\sum_i\mathbf{y}_{ij} = (1/N)\sum_{1\leq j\leq g}n_j\overline{\mathbf{y}}_{.j}$

  = mean of all cases.
- $\mathbf{E} = RCP(H_1) = \sum_j\sum_i(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{.j})(\mathbf{y}_{ij} - \overline{\mathbf{y}}_{.j})'$

  = **W** in J&W notation.

Compare these with the univariate ($p = 1$) formulas:

- $\mathbf{H} = SS_h = \sum_j n_j(\overline{y}_{.j} - \overline{y}_{..})^2$
- $\mathbf{E} = SS_e = \sum_j\sum_i(y_{ij} - \overline{y}_{.j})^2$

To get expressions for **H** and **E** from SS_h and SS_e, you replace terms of the form $(\ldots)^2$ by terms of the form $(\ldots)(\ldots)'$.

```
Cmd> manova("y=varieties",silent:T)
Cmd> list(SS)
SS              REAL   3     4     4     (labels)
Cmd> h <- matrix(SS[2,,]) # hypothesis matrix
Cmd> e <- matrix(SS[3,,] )# error matrix
Cmd> diag(h) # hypothesis SS for each variable
(1)       63.212       11.345       437.1       80.413
Cmd> diag(e) # error SS for each variable
(1)       38.956       16.962       27.223      6.1566
```

The last two lines of output are hypothesis and error SS from four univariate ANOVAs, one for each variable. You can compute F-statistics from them.

SS for a Linear Combination of Response Variables

Let $y_u \equiv \mathbf{u}'\mathbf{y} = \sum_{1\le\ell\le p} u_\ell y_\ell$ be a linear combination of <u>response variables</u>, where $\mathbf{u} = [u_\ell]_{1\le\ell\le p}$ is a vector of p weights or coefficients.

Then the N by 1 vector of all N values of y_u is

$$\mathbf{Y}_u \equiv \mathbf{Y}\mathbf{u} = \begin{bmatrix} \mathbf{y}_1'\mathbf{u} \\ \ldots \\ \mathbf{y}_N'\mathbf{u} \end{bmatrix} = \sum_{1\le\ell\le p} u_\ell \mathbf{Y}_\ell.$$

Example: $\mathbf{u}' = [1\ \ -1\ \ \ 1\ \ -1]$ for which $y_u = y_1 - y_2 + y_3 - y_4$

Facts:

The <u>univariate</u> ANOVA SS for $\mathbf{Y}_u$ are

- $SS_h(\mathbf{Y}_u) = \mathbf{u}'\mathbf{H}\mathbf{u}$, ANOVA *hypothesis* SS
- $SS_e(\mathbf{Y}_u) = \mathbf{u}'\mathbf{E}\mathbf{u}$, ANOVA *error* SS

Statistics 5401 Lecture 18 October 17, 2005

Example with $\mathbf{u}$ = [1, -1, 1, -1]'

```
Cmd> u <- vector(1,-1,1,-1)

Cmd> y_u <- y %*% u

Cmd> anova("y_u = varieties") # univariate ANOVA
Model used is y_u = varieties
             DF         SS           MS
CONSTANT      1     4284.8       4284.8
varieties     2     514.98       257.49
ERROR1      147     80.828      0.54985

Cmd> u' %*% h %*% u # SS for varieties
            (1)
(1)      514.98       varieties SS in ANOVA output

Cmd> u' %*% e %*% u # SS for error
            (1)
(1)      80.828      ERROR1 SS in ANOVA output
```

- An ANOVA consists of computing one or more hypothesis <u>sums of squares</u> SS_{h_1}, SS_{h_2}, ... and one or more error <u>sums of squares</u> SS_{e_1}, SS_{e_2}, ...
- A MANOVA consist of computing one or more hypothesis <u>matrices</u> $\mathbf{H}_1$, $\mathbf{H}_2$, ... and one or more error <u>matrices</u> $\mathbf{E}_1$, $\mathbf{E}_2$, ...

You can extract ANOVAs for all variables and of all linear combinations of variables from MANOVA **H** and **E** matrices.

Statistics 5401 Lecture 18 October 17, 2005

Comparing H and E

There are several ways.

- **Compare diagonal elements**

  $h_{\ell\ell} = SS_h(y_\ell)$ and $e_{\ell\ell} = SS_e(y_\ell)$.

  That is, say "**H** is large compared to **E**" when $\max_\ell\{h_{\ell\ell}/e_{\ell\ell}\}$ is large, or equivalently, when $\max_\ell F_\ell$ is large, where

  $$F_\ell = (h_{\ell\ell}/f_h)/(e_{\ell\ell}/f_e) = (f_e/f_h)(h_{\ell\ell}/e_{\ell\ell})$$

  are univariate F-statistics, $\ell = 1,\ldots,p$

  The critical value is $F_{f_h,f_e}(\alpha/p)$, a Bonferronized (by p) F-critical value

  This requires only *univariate* normality and constant *univariate* variances.

  When $f_h = 1$, $F = t^2$ where t is a Student's t-statistic.

Statistics 5401 Lecture 18 October 17, 2005

With **byvar:T** and **fstat:T**, **,anova()** gives all the univariate ANOVAs automatically.

```
Cmd> manova("y=varieties",byvar:T,fstat:T)
Model used is y=varieties   byvar:T => separate ANOVA tables
WARNING: summaries are sequential
                                   SepLen
               DF          SS            MS            F        P-value
CONSTANT        1      5121.7        5121.7  19326.50528       < 1e-08
varieties       2      63.212        31.606    119.26450       < 1e-08
ERROR1        147      38.956       0.26501
                                   SepWid
               DF          SS            MS            F        P-value
CONSTANT        1      1402.1        1402.1  12151.14260       < 1e-08
varieties       2      11.345        5.6725     49.16004       < 1e-08
ERROR1        147      16.962       0.11539
                                   PetLen
               DF          SS            MS            F        P-value
CONSTANT        1      2118.4        2118.4  11439.11809       < 1e-08
varieties       2       437.1        218.55   1180.16118       < 1e-08
ERROR1        147      27.223       0.18519
                                   PetWid
               DF          SS            MS            F        P-value
CONSTANT        1      215.76        215.76   5151.66322       < 1e-08
varieties       2      80.413        40.207    960.00715       < 1e-08
ERROR1        147      6.1566      0.041882
```

DF and **SS** are computed as usual.

```
Cmd> list(SS,DF)
DF              REAL    3      (labels)
SS              REAL    3      4      4      (labels)

Cmd> fh <- DF[2]; fe <- DF[3]

Cmd> h <- matrix(SS[2,,]); e <- matrix(SS[3,,]) #same as before

Cmd> fstats <- (diag(h)/fh)/(diag(e)/fe)

Cmd> fstats
(1)       119.26       49.16       1180.2       960.01
```

These match the F-statistics in the output (underlined).

To get a multivariate test, you need to Bonferronize by p.

MacAnova: Bonferronized P-values are

```
p*cumF(fstats,fh,fe,upper:T)
```

```
Cmd> 4*cumF(fstats,DF[2],DF[3], upper:T) #Bonferronized P-value
(1)  6.6787e-31  1.7968e-16  1.1427e-90  1.6678e-84
```

All are very small indicating you can reject

H_0: no treatment effect on any variable.

You can compute them directly from **H** and E by

```
p*cumF((diag(h)/fh)/(diag(e)/fe),\
     fh,fe,upper:T)
```


By analogy with the F-statistic

$$(f_e/f_h)SS_h/SS_e$$

another way to compare **H** and **E** is by the matrix "Ratio" $\mathbf{E}^{-1}\mathbf{H}$ or $(f_e/f_h)\mathbf{E}^{-1}\mathbf{H}$

- When H_0 is true, $(f_e/f_h)\mathbf{E}^{-1}\mathbf{H}$ should be "close" to $\mathbf{I}_p$ (in the same way that F should be "close" to 1).
- When H_1 is true $(f_e/f_h)\mathbf{E}^{-1}\mathbf{H}$ should be "larger" than $\mathbf{I}_p$

A test would be something like

Reject H_0 when $\mathbf{E}^{-1}\mathbf{H}$ is "too large" as compared to $(f_h/f_e)\mathbf{I}_p$, or equivalently

Reject H_0: when $(f_e/f_h)\mathbf{E}^{-1}\mathbf{H}$ is too large as compared to $\mathbf{I}_p$

Here's a problem:

$\mathbf{E}^{-1}\mathbf{H}$ is a p by p matrix. What number or numbers measure how large it is?

- $\det(\mathbf{E}^{-1}\mathbf{H})$ does **not** work as such a number because

$$\det(\mathbf{E}^{-1}\mathbf{H}) = \det(\mathbf{E}^{-1})\det(\mathbf{H}) = \det(\mathbf{H})/\det(\mathbf{E})$$

But when $f_h < p$, $\det(\mathbf{H}) = 0$, making $\det(\mathbf{E}^{-1}\mathbf{H}) = 0$ so this is *not* helpful.

What does work are measures computed from the *eigenvalues* of **H** relative to **E**, that is the *relative eigenvalues.*

See the handout for a fairly complete explanation.