Displays for Statistics 5401/8401

Lecture 9

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I did a small <u>simulation experiment</u> to accomplish two purposes:

- Examine the *power* of the Q-Q $\sqrt{\chi^2}$ <u>correlation test for normality</u>
- Check *robustness* to non-normality of the small sample Hotelling's T^2 distribution . I used M = 5000 trials, with n = 50 and p = 4.

In each table there are four lines

Intended α The significance level used in the normality test or T^2 test

Power = power of the normality test with significance level α.

α with $F_{4,46}$ = actual $\alpha \cong$ P(reject H_0: $\mu = 0$) using T^2 with <u>small n</u> critical value $((f_e p)/(f_e - p + 1))F_{p,f_e-p+1}(\alpha)$.

α with χ_4^2 = actual $\alpha \cong$ P(reject H_0: $\mu = 0$) using T^2 with <u>large n</u> critical value $\chi_4^2(\alpha)$

1. x_1, x_2, x_3, x_4 independent Student's t_5

Intended α	.01	.02	.05	.10
Power	.2282	.2902	.3982	.5042
α with $F_{4,46}$	.0068	.0162	.0446	.0996
α with χ_4^2	.0206	.0356	.0778	.1412

Distribution of t_5 has symmetric bell-shaped density but with thicker tails than normal density.

Power is moderate, small sample actual is reasonably close to intended but large sample actual α is not.

2. Independent Student's t_3, $\mu_0 = 0$.

Intended α	.01	.02	.05	.10
Power	.5560	.6328	.7490	.8250
Actual α	.0064	.0154	.0422	.0958
α with χ_4^2	.0174	.0336	.0772	.1388

t_3 is less normal than t_5 and power of correlation test is larger; actual small sample α is a little worse than for t_5.

3. Independent χ_{10}^2, μ_0 = `rep(10, 4)`.

Intended α	.01	.02	.05	.10
Power	.129	.1852	.2886	.3884
Actual α	.016	.027	.0522	.1022
α with χ_4^2	.0306	.0438	.083	.1418

χ_{10}^2 is quite skewed with mean 10.

Power is less than for t_5, actual α is not bad, a little larger than intended.

4. Independent χ_4^2, μ_0 = `rep(4, 4)`.

Intended α	.01	.02	.05	.10
Power	.3306	.4304	.5952	.7142
Actual α	.0206	.0332	.067	.1214
α with χ_4^2	.0396	.0574	.1016	.160

χ_4^2 is more skewed with mean 4.

Power is larger than for χ_{10}^2 and t_5, actual α is double the intended α for α = .10 and somewhat too large for smaller α.

Test of a vector parameter

<u>Problem</u>: Test H_0: $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, where $\boldsymbol{\theta}$ is a vector of q parameters estimated by $\hat{\boldsymbol{\theta}}$, with $V[\hat{\boldsymbol{\theta}}]$ consistently estimated by $\hat{V}[\hat{\boldsymbol{\theta}}]$.

If $\mathbf{C} = \hat{V}[\hat{\boldsymbol{\theta}}]^{1/2}$ is a square root of $\hat{V}[\hat{\boldsymbol{\theta}}]$ (i.e., $\mathbf{C}'\mathbf{C} = \hat{V}[\hat{\boldsymbol{\theta}}]$) you might hope to base a test on the multistandardized statistic

$$\mathbf{Z} = (\mathbf{C}^{-1})'(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o) = (\{\hat{V}[\hat{\boldsymbol{\theta}}]\}^{-1/2})'(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_o)$$

When $\hat{\boldsymbol{\theta}} \approx N_q(\boldsymbol{\theta}_o, V[\hat{\boldsymbol{\theta}}])$, $\mathbf{Z} \approx N_q(\mathbf{0}, \mathbf{I}_q)$.

Because $\hat{V}[\hat{\boldsymbol{\theta}}]^{1/2}$ is *not* unique, this is problematical. However, the statistic

$$T^2 = \|\mathbf{Z}\|^2 = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)'(\hat{V}[\hat{\boldsymbol{\theta}}])^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$$

is unique.

This forms the basis for many tests in both univariate and multivariate analysis.

- 1-way ANOVA F-test ($\boldsymbol{\theta} = [\mu_1,\ldots,\mu_g]'$, $\hat{\boldsymbol{\theta}} = [\overline{x}_{.1}, \overline{x}_{.2}, \ldots, \overline{x}_{.g}]'$, $\hat{V}[\hat{\boldsymbol{\theta}}] = \text{diag}\{s^2/n_i\}_{1\leq i\leq g}$
- $\chi^2 = \sum_i (O_i - E_i)^2/E_i$ goodness of fit test.

Comment: There are several statistics called <u>Hotelling's T^2</u>

- <u>one-sample</u> Hotelling's T^2 (H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$)
- <u>two-sample</u> Hotelling's T^2 (H_0: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$)
- <u>paired</u> Hotelling's T^2 (H_0: $\boldsymbol{\mu}_d = \mathbf{0}$), etc.

They may differ in

- The <u>dimension</u> p = number of means being tested.
- The <u>error degrees of freedom</u> f_e.
  f_e is usually the same as for the analogous Student's t degrees of freedom

Expressed in terms of p and f_e, *with normal data*, their <u>small sample</u> distributions are all the same:

$$T^2 = (pf_e/(f_e - p + 1))F_{p,f_e - p + 1}$$

To use this you must know 2 numbers:

- p = <u>dimension</u>. This is usually obvious.
- f_e = <u>error d.f.</u> You can usually use f_e used in univariate test.

There are special formulas for particular cases, but I recommend you *don't* use them.

Examples:

For a <u>one-sample</u> T^2, when you plug in f_e = n - 1 into

$$T^2 = \{pf_e/(f_e - p + 1)\}\ F_{p,f_e-p+1}$$

you get the "special" formula

$$T^2 = \{p(n-1)/(n-p)\}F_{p,n-p}$$

In the <u>two-sample</u> case, $f_e = n_1+n_2-2$. When you plug this into the general form you get the "special" formula

$$T^2 = \{p(n_1+n_2-2)/(n_1+n_2-p-1)\}F_{p,n_1+n_2-p-1}$$

Hotelling's one-sample

$$T^2 = (\overline{\mathbf{x}} - \boldsymbol{\mu}_0)'\hat{\mathbf{V}}(\overline{\mathbf{x}})^{-1}(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)$$

is one way to test H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

Another way is to use **Bonferronized Student's t tests**.

H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ is equivalent to the p <u>univariate</u> null hypotheses H_{0j}: $\mu_j = \mu_{0j}$. You can test each of them with a t-statistic

$$t_1 = (\overline{x}_1 - \mu_{01})/\sqrt{(s_{11}/n)},$$
$$t_2 = (\overline{x}_2 - \mu_{02})/\sqrt{(s_{22}/n)}$$
$$\ldots,$$
$$t_p = (\overline{x}_p - \mu_{02})/\sqrt{(s_{pp}/n)}$$

When H_{0j} is true t_j is Student's t on f_e = n-1 d.f.

You can reject H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ if any t_j is significantly large.

You have to be careful about how you determine which t_j are significant. If you just compare t_j with $t_{f_e}(\alpha/2)$ (two-tail tests), you will reject H_0 with probability substantially greater than α.

The solution is to *Bonferronize* the p t-tests. You do this by one of two ways:

- Use critical value $t_{f_e}((\alpha/p)/2)$
- Multiply each P-value by p

The resulting test of H_0 has true significance level $\leq \alpha$.

This is just as truly a multivariate test as is T^2.

Because (actual α) $\leq$ (intended α), this is a <u>conservative test</u>. In many cases, (actual α) $\tilde{=}$ (intended α).

MacAnova example based on Fisher *Iris setosa* data

```
Cmd> y <- read("","t11_05",quiet:T) # read JWData5.txt
Read from file "TP1:Stat5401:Data:JWData5.txt"
```

quiet:T prevents descriptive comments.

Test H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ = [5.0, 3.4, 1.4, 0.2]':

```
Cmd> setosa <- y[y[,1]==1,-1] # extract setosa flower variables

Cmd> stats <- tabs(setosa,mean:T,covar:T) # compute

Cmd> compnames(stats) # names of components
(1) "mean"          1 by p Row vector
(2) "covar"         p by p matrix

Cmd> n <- nrows(setosa) # n = 50

Cmd> p <- ncols(setosa) # 4

Cmd> xbar <- stats$mean # column vector

Cmd> vhat <- stats$covar/n # V̂[x̄] = S/n

Cmd> mu_0 <- vector(5.0,3.4,1.4,0.2) # hypothesized value

Cmd> stderrs <- sqrt(diag(vhat)) # sqrt(s[i,i]/n),i=1,...,p

Cmd> tstats <- (xbar - mu_0)/stderrs; tstats # white box
(1)     0.12036     0.52231      2.5245      3.0865

Cmd> tval(setosa - mu_0') # black box t-statistics
(1)     0.12036     0.52231      2.5245      3.0865

Cmd> fe <- n-1 # error d.f. 49

Cmd> pvalues <- p*twotailt(tstats,fe); pvalues #Bonferronized
(1)      3.6188      2.4152    0.059508    0.013314

Cmd> alpha <- .05; invstu((alpha/p)/2,fe,upper:T) #Bonf critval
(1)      2.5933          Bonferronized critical value
```


Conclusion: Since t_4 = 3.0865 > 2.5933 (or **pvalues[4]** < .05), you can reject the overall H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and the 5% level.

But you can also reject H_{04}: $\mu_4 = \mu_{04}$. Thus you have learned something about how the <u>overall</u> H_0 is untrue.

Testing using T^2

```
Cmd> tsq <- hotellval(setosa - mu_0'); tsq
(1,1)       13.616       Hotelling's T^2

Cmd> invchi(.05,p,upper:T) # upper tail probability point
(1)         9.4877         5% critical value for large n

Cmd> cumchi(tsq,p,upper:T) # upper tail probability
(1,1)    0.0086268         P-value for large n

Cmd> (fe*p/(fe - p + 1))*invF(.05, p, fe - p + 1, upper:T)
(1)         10.968         5% critical value for small n

Cmd> cumF(((fe-p+1)/(fe*p))*tsq,p,fe-p+1, upper:T)
(1,1)      0.02133         P-value for small n
```

Conclusion:
Since T^2 = 13.616 > $T_{.05}^2$ = 10.968 (or because P = .02133 < .05), you can **reject** H_0: $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

However, T^2 gives no information about how this overall H_0 is violated. This is a disadvantage of T^2 as compared to Bonferronized t-statistics.

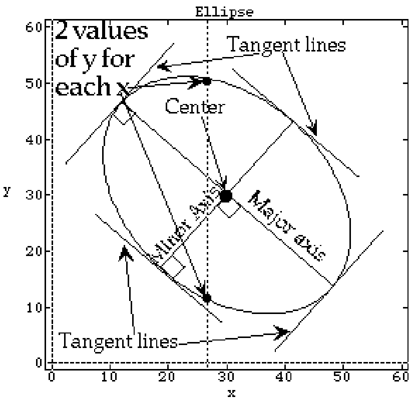
At this point it is sometimes suggested you do a "post hoc" analysis using t-statistics. But if you do that, you might as well start out with t-statistics.

Notes on MacAnova Computation

- `solve(a,b)` or `a %\% b` computes $\mathbf{A}^{-1}\mathbf{b}$ and can be more accurate than `solve(a) %*% b`.
- `cumchi(x,df)` = $P(\chi_{df}^2 \le x)$
  `cumchi(x,df,upper:T)` = $P(\chi_{df}^2 \ge x)$
  $P(\chi_{df}^2 \le$ `invchi(p,df)`$)$ = p.
  $P(\chi_{df}^2 \ge$ `invchi(p,df,upper:T)`$)$ = p
- `cumF(x,df1,df2 [,upper:T])` and `invF(p,df1,df2 [,upper:T])` compute lower tail [upper tail] <u>cumulative probabilities</u> and <u>critical values</u> for the F distribution.
- I prefer first to compute $\hat{V}$ to use in
  $$T^2 = (\overline{\mathbf{x}} - \boldsymbol{\mu}_0)'\hat{V}[\overline{\mathbf{x}}]^{-1}(\overline{\mathbf{x}} - \boldsymbol{\mu}_0),$$
  rather than use a formula like
  $$T^2 = n(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)$$
  n which $\hat{V}[\overline{\mathbf{x}}] = \mathbf{S}/n$ is sort of "hidden."

Ellipses and Ellipsoids

Here is an ellipse, a curve in the p = 2 dimensional x-y plane:



<u>Major</u> and <u>minor</u> axes are the longest and shortest "diameters" and are perpendicular to (⊥) <u>tangent lines</u>

Other diameters are not ⊥ tangent lines. All points (x,y) on an ellipse with <u>center</u> (x_0, y_0) satisfy an equation like

$$a(x-x_0)^2 + 2b(x-x_0)(y-y_0) + c(y-y_0)^2 = K^2$$

where a, b, c and K are constants with

- $a > 0$, $c > 0$ and $K > 0$
- $ac - b^2 > 0$

To get an expression for y in terms of x, you solve a <u>quadratic equation</u> to get:

$$y = f(x) \equiv y_0 - b(x-x_0)/c \pm \sqrt{\{K^2/c - (ac - b^2)((x-x_0)/c)^2\}}$$

You can't get a real square root of a negative number, so y is defined only for x that satisfy

$$K^2/c - (ac - b^2)((x-x_0)/c)^2 \ge 0,$$

that is, only for x that satisfy

$$|x-x_0| \le K\sqrt{(c/D)}, \text{ with } D = ac - b^2 > 0$$

Define q^{11}, q^{12}, q^{21}, and q^{22} by

$$q^{11} \equiv a \qquad q^{12} = q^{21} \equiv b \qquad q^{22} \equiv c$$

In terms of q^{ij}, the ellipse equation is

$$q^{11}(x-x_0)^2 + 2q^{12}(x-x_0)(y-y_0) + q^{22}(y-y_0)^2 = K^2$$

with $q^{11} > 0$, $q^{22} > 0$, $D = q^{11}q^{22} - (q^{12})^2 > 0$.

Role of the constants

- The <u>center of the ellipse</u> is at (x_0, y_0).
- For given q^{11}, q^{12} and q^{22}, the <u>size of the ellipse</u> is determined by K. A larger K produces a larger ellipse
- The <u>shape of the ellipse</u> is determined by the ratios q^{11}/K^2, q^{12}/K^2 and q^{22}/K^2.

Define the matrix

$$\mathbf{Q}^{-1} \equiv \begin{bmatrix} q^{11} & q^{12} \\ q^{12} & q^{22} \end{bmatrix}$$

Then

- the conditions
  $$q^{11} > 0 \qquad q^{22} > 0 \qquad D = q^{11}q^{22} - (q^{12})^2 > 0$$
  are completely equivalent to $\mathbf{Q}^{-1}$ being <u>positive definite</u>.
- $D = \det(\mathbf{Q}^{-1})$