Displays for Statistics 5303

Lecture 16

October 9, 2002

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Class Web Page
http://www.stat.umn.edu/~kb/classes/5303

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Non-central t

You should the non-central F distribution with numerator d.f. = 1 to find the power of a t-test of a contrast only when you plan a *two*-tail test.

Sometimes, you have a one-sided alternative to H_0: $\sum_i w_i \alpha_i = 0$:

- H_a: $\sum_i w_i \alpha_i > 0$ (reject for $t > t_\alpha$)

or

- H_a: $\sum_i w_i \alpha_i < 0$ (reject for $t < -t_\alpha$)

When H_0 is false, t has what is known as the **non-central t-distribution** on df_{error} degrees of freedom and **non-centrality parameter** $\Delta = \sqrt{n}\sum w_i \alpha_i/(\sigma\sqrt{\{\sum w_i^2\}})$.

$\Delta = 0$ corresponds to ordinary (central) t.

$t^2_{df,\Delta} = F_{1,df,\Delta^2}$, so $\Delta^2 = \zeta$.

When $\Delta \neq 0$, t does not have 0 mean and is non-symmetric about its mean.

To compute non-central t probabilities in MacAnova, you use **cumstu()** with Δ as argument 3.

Find power of 1% two-tail t-test when

$g = 6$, $n = 5$, $\sum w_i\alpha_i = D = 1.5$, $\sigma^2 = 1.26$, $w=\{1/3,1/3,1/3,-1/3,-1/3,-1/3\}$

```
Cmd> w <- vector(1,1,1,-1,-1,-1)/3; sigmasq <- 1.26

Cmd> g <- 6; n <- 5; D <- 1.5

Cmd> delta <- sqrt(n)*D/sqrt(sum(w^2)*sigmasq);delta
(1)      3.6596

Cmd> t_005 <- invstu(1 - .01/2, g*(n-1)); t_005
(1)      2.7969          two tail 1% cutpoint
```

Now compute $P(|t_{noncentral}| \geq t_{.005}) =$

$P(t_{noncentral} \leq -t_{.005}) + P(t_{noncentral} \geq +t_{.005})$

```
Cmd> cumstu(-t_005,g*(n-1), delta) +\
        1 - cumstu(t_005,g*(n-1), delta)
(1)     0.79612          Two tail power
```

This matches the power computed using **power2()**, using $\zeta = \Delta^2$.

```
Cmd> power2(delta^2, 1, .01, g*(n-1)) # power for n = 5
(1)      0.79612
```


Find *one*-tail power against the alternative $H_{a:}$ $\sum_i w_i\alpha_i = D = 1.5$

```
Cmd> t_01 <- invstu(1 - .01,g*(n-1)); t_01
(1)       2.4922        one tail 1% cutpoint

Cmd> 1 - cumstu(t_01,g*(n-1),sqrt(n)*delta1)
(1)     0.86934         one tail power
```

As you should expect, the power of the one-tail test is larger than the power of the two-tail test.

MacAnova comment

There are only two uses for which **samplesize()** gives the correct answer, both involving finding a sample size to achieve **specified power**:

(a) CRD, equal n, H_0: $\alpha_1 = \ldots = \alpha_g = 0$

(b) RCB, H_0: $\alpha_1 = \ldots = \alpha_g = 0$

It may give approximately the right sample size with other designs.

You *cannot* use **samplesize()** to find n for testing a contrast.

You can *never* use **samplesize()** to find n that has specified margin of error.

Factorial Experiments

Many experiments are designed to explore the effect of more than one categorical explanatory variables at a time.

That is, if, say, there are two categorical explanatory variables A and B, each treatment is defined by a combination of one level of A and one level of B.

Example: You have 6 diets defined by the choice of protein type (beef (B), cereal (C) or pork (P)) and whether it was high or low protein

Trt #	1	2	3	4	5	6
Level	High	High	High	Low	Low	Low
Type	B	C	P	B	C	P

This is a 2 by 3 factorial treatment structure. There are $g = 2\times3 = 6$ treatments. It is *complete* because all combinations of levels of the two factors are in the experiment.

Example: You have g =4 treatments to study the effect of adding superphosphate (P) and/or potash (K). Treatments are

1	2	3	4
None	P	K	P and K

The two categorical variables are absence or presence of P and absence or presence of K:

	1	2	3	4
P	No	Yes	No	Yes
K	No	No	Yes	Yes

Some data (n = 4) (Snedecor & Cochran):

Treat	None	P	K	P and K
	183	356	224	329
	176	300	258	283
	291	301	244	308
	254	271	214	326
Means	226.0	307.0	235.0	311.5
Vars	3119.3	1260.7	390.7	447.0

The original data was actually from a RCB, but I'm treating it like a CRD here.

Here is an ANOVA of these data, ignoring the factorial structure.

```
Cmd> y <- vector(183,176,291,254, 356,300,301,271,\
        224,258,244,214,329,283,308,326)

Cmd> treat <- factor(rep(run(4),rep(4,4)))#4 1s,4 2s,4 3s,4 4s

Cmd> anova("y=treat",fstat:T)
Model used is y=treat
              DF          SS          MS           F      P-value
CONSTANT       1  1.1653e+06  1.1653e+06   893.36504            0
treat          3       25009      8336.2     6.39079    0.0078075
ERROR1        12       15653      1304.4
```

There is strong evidence the treatments differ.

You need three subscripts to identify each response measurement

y_{ijk} = k^{th} response at levels i and j of factors A and B

Treat	P_1K_1	P_2K_1	P_1K_2	P_2K_2
	y_{111}	y_{211}	y_{121}	y_{221}
	y_{112}	y_{212}	y_{122}	y_{222}
	y_{113}	y_{213}	y_{123}	y_{223}
	y_{114}	y_{214}	y_{124}	y_{224}
Means	$\overline{y}_{11\bullet}$	$\overline{y}_{21\bullet}$	$\overline{y}_{12\bullet}$	$\overline{y}_{22\bullet}$
Vars	s_{11}^2	s_{21}^2	s_{12}^2	s_{22}^2
n_{ij}	4	4	4	4

The mean over all cases with level i of factor A is $\bar{y}_{i\bullet\bullet} = \sum_j \sum_k y_{ijk} / n_{i+}$, $n_{i+} = \sum_j n_{ij}$

The mean over all cases with level j of factor B is $\bar{y}_{\bullet j\bullet} = \sum_i \sum_k y_{ijk} / n_{+j}$, $n_{+j} = \sum_i n_{ij}$

The mean over all cases is

$$\bar{y}_{\bullet\bullet\bullet} = \sum_i \sum_j \sum_k y_{ijk} / N,\ N = n_{++} = \sum_i \sum_j n_{ij}$$

The '+' in place of subscript means sum over the subscript.

When there are a levels of A and b levels of B and all $n_{ij} = n$,

- $n_{i+} = b \times n$, all levels i of A
- $n_{+j} = a \times n$, all levels j of B
- $n_{++} = a \times b \times n$

With three factors, A, B and C, you would notate a response value as $y_{ijk\ell}$, where ℓ is the replication number for the treatment determined by levels i, j and k of factors A, B and C, respectively.

Q. What would $\bar{y}_{1\bullet 3\bullet}$ be?

A. The mean of all cases in treatments with levels A_1 and C_3, ignoring B.

The examples are examples of **complete factorial structure**. The treatments consist of all combinations of the levels of each factor. With complete a by b factorial treatments, $g = a \times b$.

Some industrial experiments may involve 20 or more factors. Even if each factor has only two levels, when there are k factors, $g = 2^k$, which can be huge. For k = 20, $g = 2^{20} = 1.05 \times 10^6$ > one million.

It is not essential to the definition of factorial structure that all combinations be present. For instance, an experiment with only treatments P_1K_2, P_2K_1 and P_2K_2 has *incomplete* factorial structure.

Certain types designs for incomplete factorial experiments are what are called **fractional factorial designs**. These are extremely important in situations in which you have many factors.

Factorial analysis is based on a particular type of model for the treatment means μ_{ij}. Specifically, with two factors, the model is

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij}$$

where $\{\alpha_i\}$ are factor A main effects, $\{\beta_j\}$ are factor B main effects and $\{\alpha\beta_{ij}\}$ are interaction effects. ($\alpha\beta_{ij}$ doesn't mean $\alpha\times\beta_{ij}$.)

Equivalently, factorial analysis is based on particular **families of contrasts** among the μ_{ij}, specifically **main effect contrasts** and **interaction contrasts**.

It's important to distinguish between a factorial treatment *structure* and a factorial *analysis*.

There are cases where you have factorial structure, but the $\{\mu_{ij}\}$ can be modeled more simply than the factorial model.

Example: 2^3 model with means like

A	1				2			
B	1		2		1		2	
C	1	2	1	2	1	2	1	2
μ	3.1	3.1	3.1	3.1	3.1	3.1	3.1	9.2

All means except μ_{222} are the same.

This has a complicated *factorial* structure involving A, B and C main effects, two-way interaction effects and three-way interaction.

But it is more simply defined in terms of two means.

Contrasts:
Here is a table of means for a 2 by 3 experiment.

	B		
A	μ_{11}	μ_{12}	μ_{13}
	μ_{21}	μ_{22}	μ_{23}

The sample means would be

	B		
A	$\overline{y}_{11}$	$\overline{y}_{12}$	$\overline{y}_{13}$
	$\overline{y}_{21}$	$\overline{y}_{22}$	$\overline{y}_{23}$

Based on sample sizes

	B		
A	n_{11}	n_{12}	n_{13}
	n_{21}	n_{22}	n_{23}

A **contrast** among the means must have a w_{ij} for each mean so you can also arrange them in a table

	B		
A	w_{11}	w_{12}	w_{13}
	w_{21}	w_{22}	w_{23}

If $\{w_{ij}\}$ a contrast, you must have $\sum_i \sum_j w_{ij} = 0$. Here is an example

	B		
A	-1	-1	1
	1	-1	1

This has three -1's and three +1's so they add to 0.

If you knew A had no effect, then this would effectively be an experiment with $g = 3$, and you might use the orthogonal contrasts {1,-1,0} and {-1,-1,2}.

	B		
A	1	-1	0
	1	-1	0

and

	B		
A	-1	-1	2
	-1	-1	2

These are **main effect contrasts** for B, ignoring A.

Similarly, for factor A, you would use the contrast {1, -1}, which translates to

	B		
A	1	1	1
	-1	-1	-1

This is a **main effect contrast** for B, ignoring A.

In general, for an A main effect contrast, the contrast weights for a given level of A are the same for all levels of B.

That is, when A has a levels, an A main effect contrast for A

	B			
A	w_1	w_1	. . .	w_1
	w_2	w_2	. . .	w_2
	. . .	. . .	. . .	. . .
	w_a	w_a		w_a

where $\sum_{1 \le i \le a} w_i = 0$

In general, for a B main effect contrast, the contrast weights for a given level of B are the same for all levels of A.

That is, when B has b levels, a B main effect contrast has the form

	B			
A	w_1	w_2	. . .	w_b
	w_1	w_2	. . .	w_b
	. . .	. . .	. . .	. . .
	w_1	w_2		w_b

where $\sum_{1 \le j \le b} w_j = 0$. The contrast weights for a given level of B are the same for all levels of A.

These are interaction contrasts

	B		
A	1	-1	0
	-1	1	0

and

	B		
A	-1	-1	2
	1	1	-2

In general, if

	B			
A	w_{11}	w_{12}	. . .	w_{1b}
	w_{12}	w_{22}	. . .	w_{2b}
	. . .	. . .	. . .	. . .
	w_{a1}	w_{a2}	. . .	w_{ab}

Where

$\sum_{1 \le i \le a} w_{ij} = 0$, $j = 1, \ldots, b$ (column sum)

$\sum_{1 \le j \le b} w_{ij} = 0$, $i = 1, \ldots, a$ (row sum)

In our 2 by 3 case, the interaction contrasts, in terms of means are

$$(\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22})$$
$$= (\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22})$$

and

$$(-\mu_{11} - \mu_{12} + 2\mu_{13}) - (\mu_{21} + \mu_{22} - 2\mu_{23})$$
$$= -(\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22}) + 2(\mu_{13} - \mu_{23})$$

These are both contrasts of contrasts.