Notation for the course

$\mathcal{U}$ is a finite population N units labeled $1, 2, \ldots, N$.
The basic population quantities.

$$y = (y_1, y_2, \ldots, y_N) = \text{the population charateristic of interest}$$

$$t(y) = t = \sum_{i=1}^{N} y_i = Y = \text{the population total}$$

$$\mu(y) = \mu = \sum_{i=1}^{N} y_i/N = Y/N = \bar{Y} = \text{the population mean}$$

$$\sigma^2(y) = \sigma^2 = \sum_{i=1}^{N} (y_i - \bar{Y})^2/(N-1) = S^2 = \text{the population variance}$$

Let smp denote the labels of the units in a sample of size n.
The basic sample quantities.

$$y_{smp} = \{y_i \,:\, i \in smp\} = \text{the observed sample values of the charteristic}$$

$$\bar{y}_{smp} = \bar{y} = \sum_{i \in smp} y_i/n = \text{the sample mean}$$

$$s^2 = \sum_{i \in smp} (y_i - \bar{y}_{smp})^2/(n-1) = \text{the sample variance}$$

Assume from a domain D of size N_d we have a sample smp_d of size n_d.

$$t_d(y) = t_d = \sum_{i \in D} y_i = Y_d = \text{the domain total}$$

$$\mu_d(y) = \mu_d = \sum_{i \in D} y_i/N_d = \text{ the domain mean}$$

$$\sigma_d^2(y) = \sigma_d^2 = \sum_{i \in D} (y_i - \mu_d(y))^2/(N_d - 1) = \text{the domain variance.}$$

$$\bar{y}_{smp_d} = \bar{y}_d = \sum_{i \in smp_d} y_i/n_d = \text{the domain sample mean}$$

$$s_d^2 = \sum_{i \in smp_d} (y_i - \bar{y}_d)^2/(n_d - 1) = \text{the domain sample variance}$$

Suppose the population has H strata where stratum h contains N_h units with $\sum_{h=1}^{H} N_h = N$. Let smp_h be the labels of the n_h units in the sample that belong to stratum h. Here $\sum_{h=1}^{H} h_h = n$. Notation for the basic population and sample quantities follow.

$$\begin{aligned}
y_{hi} &= \text{value of the } i\text{th unit in stratum } h \\
t_h &= \sum_{i=1}^{N_h} y_{hi} = \text{total for stratum } h \\
t &= \sum_{i=1}^{N} t_h = \text{population total} \\
\mu_h &= t_h/N_h = \bar{Y}_h = \text{mean for stratum } h \\
\mu(y) = \mu &= \sum_{h=1}^{H}\sum_{i=1}^{N_h} y_{hi}/N = t/N = \bar{Y} = \text{population mean} \\
\sigma_h^2 &= \sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^2/(N_h - 1) = \text{variance for stratum } h \\
\bar{y}_{smp_h} = \bar{y}_h &= \sum_{i \in smp_h} y_{hi}/n_h = \text{sample mean for stratum } h \\
s_{smp_h}^2 = s_h^2 &= \sum_{iinsmp_h} (y_{hi} - \bar{y}_h)^2/(n_h - 1) = \text{sample variance for stratum } h
\end{aligned}$$

If we let $W_h = N_h/N$ then we can write

$$\mu = \sum_{h=1}^{H}\sum_{i=1}^{N_h} y_{hi}/N = \sum_{h=1}^{H} \frac{N_h}{N}\bar{Y}_h = \sum_h W_h \bar{Y}_h$$

whose natural estimate under simple random sampling is

$$\bar{y}_{str} = \sum_h W_h \bar{y}_h$$

Next we consider a cluster population with N cluster each of size M. We will select n clusters using simple random sampling and then within the selected clusters we independently use simple random sampling to select samples of size m. We let smp denote the labels of the n clusters in first stage sample and smp_i denote the labels of elements in second stage sample drawn from cluster i. Notation for the basic population and sample quantities follow.

$$Y_i = t_i = \sum_{j=1}^{M} y_{ij} = \text{total for } i\text{th cluster; } Y_i/M = \bar{Y}_i = \text{mean for } i\text{th cluster}$$

$$Y = \sum_{i=1}^{N} Y_i = \text{population total}$$

$$\bar{Y} = Y/N = \text{population mean of cluster totals}$$

$$\bar{\bar{Y}} = \sum_{i=1}^{N}\sum_{j=1}^{M} y_{ij}/NM = \text{population mean of elements}$$

$$\sigma_t^2 = \sum_{i=1}^{N}(Y_i - \bar{Y})^2/(N-1) = \text{population variance of cluster totals}$$

$$\sigma^2 = \sum_{i=1}^{N}\sum_{j=1}^{M}(y_{ij} - \bar{\bar{Y}})^2/(NM-1) = \text{population variance of elements}$$

$$\sigma_i^2 = \sum_{j=1}^{M}(y_{ij} - \bar{Y}_i)^2/(M-1) = \text{ variance within cluster } i$$

$$\sigma_w^2 = \sum_{i=1}^{N}\sigma_i^2/N = \text{over all measure of variability within clusters}$$

$$\sigma_b^2 = \sum_{i=1}^{N}(\bar{Y}_i - \bar{\bar{Y}})^2/(N-1) = \text{measure of variability between clusters}$$

$$\bar{y}_{smp_i} = \bar{y}_i = \sum_{j\in smp_i} y_{ij}/m = \text{ mean of elements in sample for cluster } i$$

$$\bar{\bar{y}}_{smp} = \bar{\bar{y}} = \sum_{i\in smp} \bar{y}_i/n = \text{sample mean of all the elements in the sample}$$

$$s_t^2 = \sum_{i\in smp}\left(M_i\bar{y}_i - \sum_{i\in smp} M_i\bar{y}_i/n\right)^2/(n-1) = \text{sample estimate of } \sigma_t^2$$

$$s_w^2 = \sum_{i\in smp}\sum_{j\in smp_i}(y_{ij} - \bar{y}_i)^2/(n(m-1) = \text{within cluster variation for the sample}$$

$$s_b^2 = \sum_{i\in smp}(\bar{y}_i - \bar{\bar{y}})^2/(n-1) = \text{between cluster variability for clusters in sample}$$

Note it is easy to check that $\sigma_b^2 = \sigma_t^2/M^2$