Statistics 5401
25. Canonical Correlation

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Factor analysis helps us model and summarize correlation in a set of variables. What should we do if we want to summarize correlation between two sets of variables?

For example, suppose that $x^{(1)}$ is a vector of eight components related to wet (acidic) deposition at a site: pH and the concentrations of SO_4, NO_3, Cl, NH_4, Ca, Na, and Mg. Also suppose that $x^{(2)}$ is vector of three components related to dry deposition: SO_2, O_3, PM10.

What aspects of $x^{(1)}$ are most correlated with $x^{(2)}$?

That's a pretty general question. Usually we settle for what linear combinations of $x^{(1)}$ and $x^{(2)}$. That is we want to find the linear combinations of $x^{(1)}$ and $x^{(2)}$ that are most highly correlated, or find a, b such that

$$\rho = cor(a'x^{(1)}, b'x^{(2)})$$

is maximized.

ρ is called a *canonical correlation*, and $u = a'x^{(1)}$ and $v = b'x^{(2)}$ are a canonical variable pair.

OK, let's state the set up formally. $x^{(1)}$ is a random p-vector, and $x^{(2)}$ is a random q-vector. For simplicity, assume that $p \le q$.

Let $a_i, i = 1, 2, \ldots, p$ be orthonormal p-vectors, and let $b_i, i = 1, 2, \ldots, p$ be orthonormal q-vectors.

Choose a_1 and b_1 such that $\rho_1 = cor(a_1'x^{(1)}, b_1'x^{(2)})$ is maximized.

Choose a_k and b_k such that $\rho_k = cor(a_k'x^{(1)}, b_k'x^{(2)})$ is maximized, subject to the pairs $a_i'x^{(1)}, a_k'x^{(1)}$ and $b_i'x^{(2)}, b_k'x^{(2)}$ being uncorrelated for $1 \le i < k$.

The ρ_is are the (population) canonical correlations, and the $u_i = a_i'x^{(1)}$, $v_i = b_i'x^{(2)}$ are (population) canonical variable pairs.

There are at most p nonzero canonical correlations, regardless of how big q is.

$x^{(1)}$ has variance Σ_{11}, $x^{(2)}$ has variance Σ_{22}.

Cov($x^{(1)}$,$x^{(2)}$) = Σ_{12}, and

$$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \qquad Var(x) = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Then

$$cor(a_1'x^{(1)}, b_1'x^{(2)}) = \frac{a_1'\Sigma_{12}b_1}{\sqrt{a_1'\Sigma_{11}a_1 \quad b_1'\Sigma_{22}b_1}}$$

Let $\tilde{x}^{(1)} = \mathbf{G}_1 x^{(1)}$ where $\mathbf{G}_1$ is a $p \times p$ invertible matrix. Similarly, $\tilde{x}^{(2)} = \mathbf{G}_2 x^{(2)}$ where $\mathbf{G}_2$ is a $q \times q$ invertible matrix. Then

$$Var(\tilde{x}^{(1)}) = \mathbf{G}_1\Sigma_{11}\mathbf{G}_1'$$
$$Var(\tilde{x}^{(2)}) = \mathbf{G}_2\Sigma_{21}\mathbf{G}_2'$$
$$Cov(\tilde{x}^{(1)}, \tilde{x}^{(2)}) = \mathbf{G}_1\Sigma_{12}\mathbf{G}_2'$$

For square matrix $\mathbf{A}$, let $\mathbf{A}^{-T} = (\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$.

Let $\tilde{a}_1 = \mathbf{G}_1^{-T} a_1$ and $\tilde{b}_1 = \mathbf{G}_2^{-T} b_1$. Then

$$
\begin{aligned}
& cor(\tilde{a}_1' \tilde{x}^{(1)}, \tilde{b}_1' \tilde{x}^{(2)}) \\
= & \frac{\tilde{a}_1' \mathbf{G}_1 \Sigma_{12} \mathbf{G}_2' \tilde{b}_1}{\sqrt{\tilde{a}_1' \mathbf{G}_1 \Sigma_{11} \mathbf{G}_1' \tilde{a}_1 \;\; \tilde{b}_2' \mathbf{G}_2 \Sigma_{22} \mathbf{G}_2' \tilde{b}_2}} \\
= & \frac{a_1' \mathbf{G}_1^{-1} \mathbf{G}_1 \Sigma_{12} \mathbf{G}_2' \mathbf{G}_2^{-T} b_1}{\sqrt{a_1' \mathbf{G}_1^{-1} \mathbf{G}_1 \Sigma_{11} \mathbf{G}_1' \mathbf{G}_1^{-T} a_1 \;\; b_2' \mathbf{G}_2^{-1} \mathbf{G}_2 \Sigma_{22} \mathbf{G}_2' \mathbf{G}_2^{-T} b_2}} \\
= & \frac{a_1' \Sigma_{12}^{-T} b_1}{\sqrt{a_1' \Sigma_{11} a_1 \;\; b_2' \Sigma_{22} b_2}} \\
= & cor(a_1' x^{(1)}, b_1' x^{(2)})
\end{aligned}
$$

This last results says that invertible linear combinations of the original variables do not intrinsically, change the canonical variables, because we can recover the original canonical variables from the new ones and vice versa.
Suppose that

$$\Sigma_{11} = \Sigma_{11}^{1/2} (\Sigma_{11}^{1/2})' \qquad \Sigma_{22} = \Sigma_{22}^{1/2} (\Sigma_{22}^{1/2})'$$

Let $\mathbf{G}_1 = (\Sigma_{11}^{1/2})^{-1} = \Sigma_{11}^{-1/2}$ and $\mathbf{G}_2 = (\Sigma_{22}^{1/2})^{-1} = \Sigma_{22}^{-1/2}$
Then

$$
\begin{aligned}
& Var \begin{bmatrix} \tilde{x}^{(1)} \\ \tilde{x}^{(2)} \end{bmatrix} \\
= & \begin{bmatrix} \Sigma_{11}^{-1/2} \Sigma_{11} (\Sigma_{11}^{-1/2})' & \Sigma_{11}^{-1/2} \Sigma_{12} (\Sigma_{22}^{-1/2})' \\ \Sigma_{22}^{-1/2} \Sigma_{21} (\Sigma_{11}^{-1/2})' & \Sigma_{22}^{-1/2} \Sigma_{22} (\Sigma_{22}^{-1/2})' \end{bmatrix} \\
= & \begin{bmatrix} \mathbf{I}_p & \Sigma_{11}^{-1/2} \Sigma_{12} (\Sigma_{22}^{-1/2})' \\ \Sigma_{22}^{-1/2} \Sigma_{21} (\Sigma_{11}^{-1/2})' & \mathbf{I}_q \end{bmatrix}
\end{aligned}
$$

If we choose $||\tilde{a}_1|| = 1$ and $||\tilde{b}_1|| = 1$, then

$$Var(\tilde{a}_1' \tilde{x}^{(1)}) = \tilde{a}_1' \mathbf{I} \tilde{a}_1 = 1$$

and

$$Var(\tilde{b}_1' \tilde{x}^{(2)}) = \tilde{b}_1' \mathbf{I} \tilde{b}_1 = 1$$

So with unit length $\tilde{a}_1$ and $\tilde{b}_1$, the maximal correlation occurs when

$$\tilde{a}_1' \Sigma_{11}^{-1/2} \Sigma_{12} (\Sigma_{22}^{-1/2})' \tilde{b}_1$$

is maximized.
This is just doing SVD!
Do the SVD

$$\Sigma_{11}^{-1/2} \Sigma_{12} (\Sigma_{22}^{-1/2})' = \mathbf{U} \mathbf{D} \mathbf{V}'$$

The singular values in $\mathbf{D}$ are the canonical correlations ρ_i, and the columns of $\mathbf{U}$ and $\mathbf{V}$ give the coefficents $\tilde{a}$ and $\tilde{b}$:

$$\check{\mathbf{U}}_i = \tilde{a}_i \qquad \check{\mathbf{V}}_i = \tilde{b}_i$$

We get the originals via

$$a_i = (\Sigma_{11}^{-1/2})'\tilde{a}_i \quad b_i = (\Sigma_{22}^{-1/2})'\tilde{b}_i$$

Note that for $i \neq j$

$$0 = \tilde{a}_i'\tilde{a}_j = a_i'\Sigma_{11}a_j$$

Thus the orthogonality of $\tilde{a}_i$ and $\tilde{a}_j$ gives us that $a_i'x^{(1)}$ and $a_j'x^{(1)}$ are uncorrelated, as required for canonical correlation.
Similarly, $b_i'x^{(2)}$ and $b_j'x^{(2)}$ are uncorrelated.
From the properties of SVD, $\tilde{a}_1$ is an eigenvector that satisfies

$$\rho_1^2\tilde{a}_1 = \Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-T/2}\tilde{a}_1$$

or

$$\rho_1^2(\Sigma_{11}^{1/2})'a_1 = \Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-T/2}(\Sigma_{11}^{1/2})'a_1$$

or

$$\rho_1^2a_1 = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}a_1$$

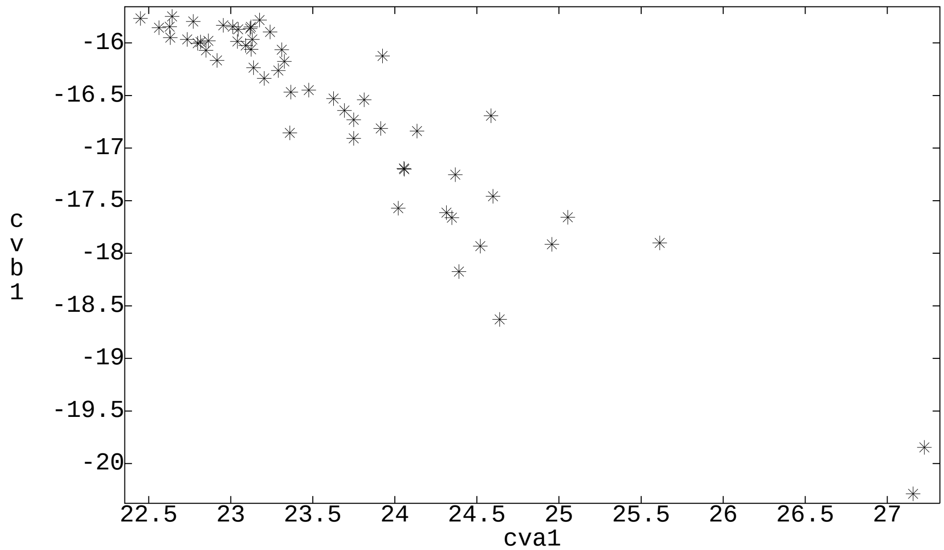
Thus a_1 is an eigenvector of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, or, alternatively, an eigenvector of $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ *relative* to Σ_{11}.
Similarly, b_1 is an eigenvector of $\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, or, alternatively, an eigenvector of $\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$ *relative* to Σ_{22}.
Let $\mathbf{A}$ be the matrix with ith row a_i', and similarly for $\mathbf{B}$. Then, for $p = 3, q = 4$

$$Cor\begin{bmatrix}\mathbf{A}x^{(1)}\\ \mathbf{B}x^{(2)}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \rho_1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & \rho_2 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & \rho_3 & 0\\ \rho_1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & \rho_2 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \rho_3 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notes.
1. Canonical correlation linear combinations summarize correlation between $x^{(1)}$ and $x^{(2)}$; they may do a poor job of summarizing the variance in $x^{(1)}$ or $x^{(2)}$.
2. The first canonical correlation is greater than the correlation between any pair of variables in $x^{(1)}$ and $x^{(2)}$.
3. When $p = 1$, the square of the canonical correlation is the R^2 for regressing $x^{(1)}$ on $x^{(2)}$.
Sample Canonical Correlations follow the same pattern, except that we work with $\mathbf{S}$ instead of Σ.

```
Cmd> dim(X)
(1)          55            8

Cmd> S <- tabs(X,covar:T)

Cmd> S11 <- S[run(5),run(5)]

Cmd> S12 <- S[run(5),run(6,8)]

Cmd> S22 <- S[run(6,8),run(6,8)]
```

```
Cmd> releigen(S12%*%solve(S22)%*%S12',S11)
component: values
(1)   0.88685  0.095624  0.018179  0  0
component: vectors
(1,1) -0.085219    4.4161     -6.01   0.95589    -2.577
(2,1)  -0.13291   0.12103    4.5718  -0.16935 -0.005325
(3,1)  0.057188   -0.1433  -0.11629   -0.3509     1.688
(4,1)    1.4605   -17.766  -0.87488    41.231   -7.1477
(5,1)    5.9672   0.68966   -3.7538   -13.833   -6.6239

Cmd> releigen(S12'%*%solve(S11)%*%S12,S22)
component: values
(1)      0.88685     0.095624     0.018179
component: vectors
(1,1)     -0.49934     -0.11488       5.6404
(2,1)     -0.41843       1.5929      -2.1826
(3,1)     0.017341      -0.3247    -0.062924

Cmd> A <- releigen(S12%*%solve(S22)%*%S12',S11)$vectors

Cmd> B <- releigen(S12'%*%solve(S11)%*%S12,S22)$vectors

Cmd> cva1 <- X[,run(5)]%*%A[,1]

Cmd> cvb1 <- X[,run(6,8)]%*%B[,1]

Cmd> cva <- X[,run(5)]%*%A

Cmd> cvb <- X[,run(6,8)]%*%B

Cmd> print(cor(cva,cvb),format:"f6.3")
MATRIX:
(1,1)  1.000  0.000  0.000 -0.000  0.000 -0.942 -0.000  0.000
(2,1)  0.000  1.000 -0.000  0.000 -0.000 -0.000  0.309  0.000
(3,1)  0.000 -0.000  1.000 -0.000  0.000 -0.000 -0.000  0.135
(4,1) -0.000  0.000 -0.000  1.000 -0.000 -0.000  0.000  0.000
(5,1)  0.000 -0.000  0.000 -0.000  1.000  0.000 -0.000  0.000
(6,1) -0.942 -0.000 -0.000 -0.000  0.000  1.000  0.000 -0.000
(7,1) -0.000  0.309 -0.000  0.000 -0.000  0.000  1.000  0.000
(8,1)  0.000  0.000  0.135  0.000  0.000 -0.000  0.000  1.000


Cmd> vector(.942,.309,.135)^2
(1)      0.88736     0.095481     0.018225

Cmd> plot(cva1,cvb1,file:"ccplot1.ps")
```




```
Cmd> readdata("",sg,sp,nas,ct,mrt,art,mt)
Read from file "˜/5401/JW5data/T9-12.DAT"
Column 1 saved as REAL vector sg
Column 2 saved as REAL vector sp
Column 3 saved as REAL vector nas
Column 4 saved as REAL vector ct
Column 5 saved as REAL vector mrt
Column 6 saved as REAL vector art
Column 7 saved as REAL vector mt

Cmd> X <- hconcat(sg,sp,nas,ct,mrt,art,mt)

Cmd> S <- tabs(X,covar:T)

Cmd> S11 <- S[run(3),run(3)]

Cmd> S12 <- S[run(3),run(4,7)]

Cmd> S22 <- S[run(4,7),run(4,7)]

Cmd> releigen(S12%*%solve(S22)%*%S12',S11)
component: values
(1)        0.989      0.77107      0.14715
component: vectors
(1,1)     0.062378      0.17407     -0.37715
(2,1)     0.020926     -0.24216      0.10352
(3,1)     0.078258      0.23829      0.38342

Cmd> releigen(S12'%*%solve(S11)%*%S12,S22)
```

```
component: values
(1)      0.989   0.77107   0.14715      0
component: vectors
(1,1)  0.069748   0.19239  -0.24656 -0.018706
(2,1)  0.030738  -0.20157    0.1419  -0.33338
(3,1)  0.089564   0.49576   0.28022 -0.052847
(4,1)   0.06283 -0.068316 -0.011333  0.093897

Cmd> A <- releigen(S12%*%solve(S22)%*%S12’,S11)$vectors

Cmd> B <- releigen(S12’%*%solve(S11)%*%S12,S22)$vectors

Cmd> cva <- X[,run(3)]%*%A

Cmd> cvb <- X[,run(4,7)]%*%B

Cmd> print(cor(cva,cvb)[run(3),run(4,7)],format:"f7.4")
MATRIX:
(1,1)  0.9945  0.0000 -0.0000 -0.0000
(2,1) -0.0000  0.8781 -0.0000 -0.0000
(3,1) -0.0000  0.0000 -0.3836  0.0000

Cmd> cor(cva,X[,run(3)])[run(3),run(4,6)]
(1,1)      0.97988      0.94641      0.95186
(2,1)  -0.00064779     -0.32288       0.1863
(3,1)      -0.1996    0.0075044      0.24341

Cmd> cor(cvb,X[,run(4,7)])[run(3),run(5,8)]
(1,1)   0.63833   0.72116   0.64725   0.94409
(2,1)    0.2157  -0.23756   0.50133  -0.19753
(3,1)  -0.65141  0.067738   0.57422  0.094226
```