Statistics 5401
12. Simultaneous Confidence

Gary W. Oehlert
School of Statistics
313B Ford Hall
612-625-1557
gary@stat.umn.edu

Let $\mathbf{X}$ be a data matrix with $\vec{\mathbf{X}}_i$ iid $\mathrm{N}_p(\mu, \Sigma)$. $\overline{\mathbf{x}}$ and S are the usual sample mean and variance.
We use T^2 to construct a $1-\alpha$ confidence region for μ as

$$\left\{\mu : n(\overline{\mathbf{x}} - \mu)'\mathbf{S}^{-1}(\overline{\mathbf{x}} - \mu) \le \frac{(n-1)p}{(n-p)} F_{\alpha,\, p,\, n-p}\right\}$$

This confidence region is an ellipsoid centered at $\overline{\mathbf{x}}$ with axes oriented along the eigenvectors of $\mathbf{S}$ and axis lengths proportional to the square roots of the eigenvalues of $\mathbf{S}$.
Let a be a p-vector, and suppose that we desire a confidence interval for $a'\mu$.

$$a'\overline{\mathbf{x}} \sim \mathrm{N}(a'\mu, a'\Sigma a/n)$$

We can construct the confidence interval

$$a'\overline{\mathbf{x}} \pm t_{\alpha/2,\, n-1}\sqrt{a'\mathbf{S}a/n}$$

by applying standard univariate methods to $\mathbf{X}a$.
In particular, if a is all zeros except for a single 1, then these confidence intervals are ordinary confidence intervals for variable means.
What if we have several p-vectors $a_1, a_2, \ldots, a_k$?
Then we can construct k intervals, each with $1-\alpha$ coverage by repeating the previous procedure.
If we want *simultaneous* intervals, we can Bonferronize k intervals by using intervals with coverage $1-\alpha/k$.

```
Cmd> readdata("",x1,x2,x3,x4,x5)
Read from file "/HOME/faculty/gary/classes/5401/JW5data/T4-3.DAT"
Column 1 saved as REAL vector x1
Column 2 saved as REAL vector x2
Column 3 saved as REAL vector x3
Column 4 saved as REAL vector x4
Column 5 saved as REAL vector x5

Cmd> X <- hconcat(x1,x2,x3,x4)

Cmd> xbar <- tabs(X,mean:T)

Cmd> S <- tabs(X,covar:T)

Cmd> tint(x1,1-.05)
(1)       1784.7       2027.5
```

```
Cmd> tint(x1,1-.05/4)
(1)       1748.1       2064.1

Cmd> tint(x2,1-.05/4)
(1)       1594.6       1904.4

Cmd> tint(x3,1-.05/4)
(1)       1361.7       1656.5

Cmd> tint(x4,1-.05/4)
(1)         1568       1881.9

Cmd> xbar+vector(-1,1)'*\
invstu(1-.05/8,29)*sqrt(diag(S)/30)
(1,1)       1748.1       2064.1
(2,1)       1594.6       1904.4
(3,1)       1361.7       1656.5
(4,1)         1568       1881.9
```

Now what if we want simultaneous coverage for *all possible* vectors a?
Ordinary t^2 for a given a is

$$t^2 = \frac{n(a'\overline{\mathbf{x}} - a'\mu)^2}{a'\mathbf{S}a} = \frac{n(a'(\overline{\mathbf{x}} - \mu))^2}{a'\mathbf{S}a}$$

What a gives us the maximum possible t^2.
Maximization lemma says $a = \mathbf{S}^{-1}(\overline{\mathbf{x}} - \mu)$. Plugging in, the maximum t^2 is

$$n(\overline{\mathbf{x}} - \mu)'\mathbf{S}^{-1}(\overline{\mathbf{x}} - \mu) = T^2$$

What this means is that if we use a multiplier big enough to control for T^2, then we get simultaneous coverage for all linear combinations.

$$a'\overline{\mathbf{x}} \pm \sqrt{\frac{p(n-1)}{(n-p)} F_{\alpha,\, p,\, n-p}\; a'\mathbf{S}a/n}$$

(Those of you who have taken design of experiments may wish to compare this with the Scheffé method.)

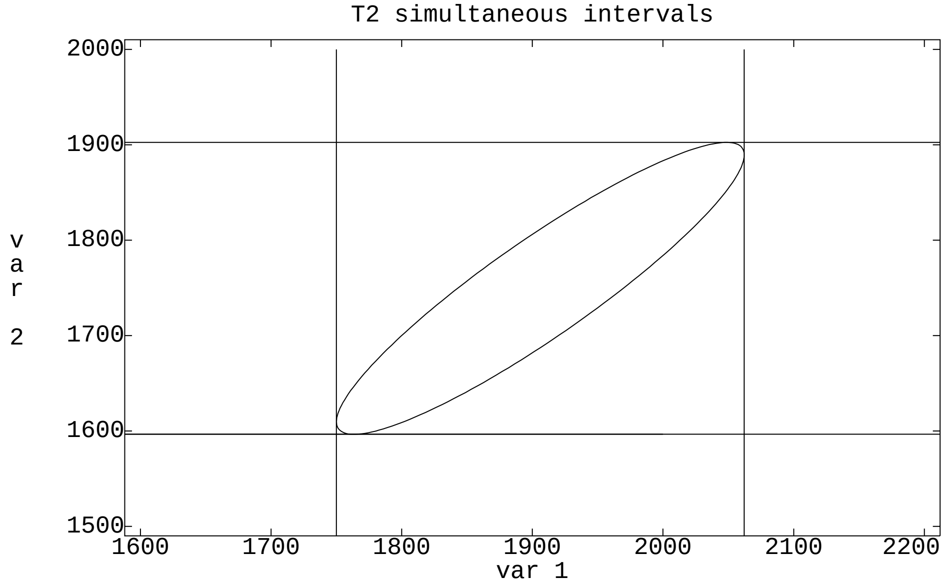
```
Cmd> xbar
(1)   1906.1   1749.5   1509.1     1725

Cmd> a <- vector(1,0,-1,0)

Cmd> a'%*%xbar + vector(-1,1)*\
sqrt(a'%*%S%*%a/30*4*29/26*invF(.95,4,26))
(1,1)       300.21
(2,1)       493.73
```

```
Cmd> xbar+vector(-1,1)'*\
sqrt(diag(S)/30*4*29/26*invF(.95,4,26))
(1,1)       1698.5      2113.7
(2,1)       1546.1        1953
(3,1)       1315.5      1702.8
(4,1)       1518.8      1931.2
```

If we look at a vectors that are all 0 except for a single 1, we get the simultaneous confidence approach applied to the individual variables.
These confidence intervals are the shadows or projections of the confidence ellipse onto the coordinate axes.



Now if all you're going to use the simultaneous intervals for is to make simultaneous intervals for the component means, you're better off using Bonferronized individual intervals, because they are shorter.