Statistics 5041
3. Matrix Multiplication

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Two matrices, $\mathbf{X}$ and $\mathbf{Y}$. $\mathbf{X}$ has dimension $a \times b$, and $\mathbf{Y}$ has dimension $b \times c$. Then we can form the *matrix product* (ie, us *matrix multiplication*)

$$\mathbf{Z} = \mathbf{X}\mathbf{Y}$$

$\mathbf{Z}$ is $a \times c$.
Note, $\mathbf{Y}\mathbf{X}$ may not be possible (dimension mismatch), and even if possible, $\mathbf{X}\mathbf{Y}$ need not equal $\mathbf{Y}\mathbf{X}$.

$$z_{ij} = \sum_{k=1}^{b} x_{ik} y_{kj}$$

Multiply corresponding elements of ith row of $\mathbf{X}$ with jth column of $\mathbf{Y}$, then add up.

$$\mathbf{X} = \begin{bmatrix} 6 & 2 & 8 \\ 8 & 5 & 6 \\ 1 & 2 & 3 \\ 9 & 4 & 5 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 7 & 3 \end{bmatrix}$$

$$\begin{aligned} z_{11} &= 6 \times 2 + 2 \times 1 + 8 \times 7 = 70 \\ z_{21} &= 8 \times 2 + 5 \times 1 + 6 \times 7 = 63 \\ z_{22} &= 8 \times 5 + 5 \times 4 + 6 \times 3 = 78 \\ \ldots \end{aligned}$$

```
Cmd> X%*%Y
(1,1)           70            62
(2,1)           63            78
(3,1)           25            22
(4,1)           57            76

Cmd> Y%*%X
ERROR: Dimension mismatch: 3 by 2 %*% 4 by 3 near Y%*%X
```

The i, j element of $\mathbf{X}\mathbf{Y}$ is the matrix product of the ith row a $\mathbf{X}$ (the ith row is a $1 \times b$ matrix) and the jth column of $\mathbf{Y}$ (the jth column is a $b \times 1$ matrix).
Let $\check{\mathbf{X}}_j$ be the jth column of $\mathbf{X}$.

$$\mathbf{X} = \left[\check{\mathbf{X}}_1 : \check{\mathbf{X}}_2 : \ldots : \check{\mathbf{X}}_p\right]$$

Let $\vec{\mathbf{X}}_i'$ be the ith row of $\mathbf{X}$.

$$\mathbf{X} = \begin{bmatrix} \vec{\mathbf{X}}_1' \\ \vec{\mathbf{X}}_2' \\ \vdots \\ \vec{\mathbf{X}}_n' \end{bmatrix} = \left[\vec{\mathbf{X}}_1 : \vec{\mathbf{X}}_2 : \ldots : \vec{\mathbf{X}}_n\right]'$$

Note: $\vec{\mathbf{X}}_i$ is a column vector.

Inner Products Let x and y be two vectors with the same length n. Then the inner product of x and y is the scalar

$$< x, y >= \sum_{i=1}^{n} x_i y_i = x'y$$

The i, j element of $\mathbf{XY}$ sums the cross products of the ith row of $\mathbf{X}$ and the jth column of $\mathbf{Y}$; that is,

$$\mathbf{Z}_{ij} = (\mathbf{XY})_{ij} = \vec{\mathbf{X}}_i'\check{\mathbf{Y}}_j =< \vec{\mathbf{X}}_i, \check{\mathbf{Y}}_j >$$

Sums of Squares and Products Suppose that $\mathbf{X}$ is an $n \times p$ matrix of data, n cases on p variables.

$$\mathbf{X}'\mathbf{X} = \left[\check{\mathbf{X}}_i'\check{\mathbf{X}}_j\right]_{1 \le i,j \le p}$$

Each element of $\mathbf{X}'\mathbf{X}$ is an inner product of two columns of $\mathbf{X}$. Diagonal elements of $\mathbf{X}'\mathbf{X}$ are

$$\check{\mathbf{X}}_j'\check{\mathbf{X}}_j = \sum_{k=1}^{n} X_{kj}^2$$

a sum of squares.

Off diagonal elements are

$$\check{\mathbf{X}}_i'\check{\mathbf{X}}_j = \sum_{k=1}^{n} X_{ki} X_{kj}$$

a sum of products.

All together, *SSP* matrix.

Matrices of this type are the basis of variance/covariance matrices and will appear throughout the course.

Outer Products Let x and y be two vectors with lengths n and m. Then the outer product of x and y is an $n \times m$ matrix

$$xy' = \begin{bmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_m \\ x_2y_1 & x_2y_2 & \ldots & x_2y_m \\ \vdots & \vdots & & \vdots \\ x_ny_1 & x_ny_2 & \ldots & x_ny_m \end{bmatrix}$$

We can also write a matrix multiplication as a sum of outer products (here, $\mathbf{X}$ is $a \times b$ and $\mathbf{Y}$ is $b \times c$).

$$\mathbf{XY} = \check{\mathbf{X}}_1\vec{\mathbf{Y}}_1 + \check{\mathbf{X}}_2\vec{\mathbf{Y}}_2 + \ldots + \check{\mathbf{X}}_b\vec{\mathbf{Y}}_b$$

```
Cmd> X[,1]%*%Y[1,]+X[,2]%*%Y[2,]+X[,3]%*%Y[3,]
(1,1)           70            62
(2,1)           63            78
(3,1)           25            22
(4,1)           57            76
```

Two MacAnova shortcuts

`X %c% Y` is $\mathbf{X}'\mathbf{Y}$.

`X %C% Y` is $\mathbf{XY}'$.