Statistics 5041
2. Matrix Manipulation

Gary W. Oehlert
School of Statistics
313B Ford Hall
612-625-1557
gary@stat.umn.edu

A *matrix* is a rectangular array of numbers.

$$\mathbf{X} = \begin{bmatrix} 6 & 2 & 8 \\ 8 & 5 & 6 \\ 1 & 2 & 3 \\ 9 & 4 & 5 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix}$$

The first subscript on x_{ij} is the row number, whereas the second subscript is the column number. $x_{23} = 6$ is the second row, third column.

```
Cmd> data <- vector(6,8,1,9,2,5,2,4,8,6,3,5)

Cmd> X <- matrix(data,4);X
(1,1)       6       2       8
(2,1)       8       5       6
(3,1)       1       2       3
(4,1)       9       4       5

Cmd> data <- enter(6 8 1 9 2 5 2 4 8 6 3 5);\
data
 (1)     6     8     1     9     2
 (6)     5     2     4     8     6
(11)     3     5

Cmd> data2 <- enter(data data)
WARNING: nonnumeric character(s) in string variable ignored
WARNING: no data found by vecread() in macro enter

Cmd> data2 <- vector(data,data)
```

`enter()` lets you skip the commas, but only accepts numeric input. `vector()` is more flexible.
A *column vector* is a matrix with just one column.

```
Cmd> X[,2] # second column
(1,1)            2
(2,1)            5
(3,1)            2
(4,1)            4
```

A *row vector* is a matrix with just one row.

```
Cmd> X[3,] # third row
(1,1)       1       2       3
```

A plain *vector* is a column vector.

```
Cmd> vector(12,5,2,4)
(1)   12    5    2    4
```

The *dimensions* of a matrix are its numbers of rows and columns.

```
Cmd> dim(X);dim(X[3,]);dim(X[,2]);\
dim(vector(12,5,2,4))
(1)             4             3
(1)             1             3
(1)             4             1
(1)             4
```

MacAnova distinguishes between a matrix with a single column (which has two dimensions), and a simple vector (which only has a single dimension). Mathematically they are the same, and they can usually be used interchangeably in MacAnova.

```
Cmd> nrows(X);nrows(X[,2]);nrows(X[3,]);\
nrows(data)
(1)             4
(1)             4
(1)             1
(1)            12
```

```
Cmd> ncols(X);ncols(X[,2]);ncols(X[3,]);\
ncols(data)
(1)             3
(1)             1
(1)             3
(1)             1
```

The *transpose* of matrix $\mathbf{X}$ (denoted $\mathbf{X}'$) has the same elements as $\mathbf{X}$ with the rows and columns interchanged.

$$\mathbf{X} = \begin{bmatrix} 6 & 2 & 8 \\ 8 & 5 & 6 \\ 1 & 2 & 3 \\ 9 & 4 & 5 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{bmatrix}$$

$$\mathbf{X}' = \begin{bmatrix} 6 & 8 & 1 & 9 \\ 2 & 5 & 2 & 4 \\ 8 & 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} & x_{31} & x_{41} \\ x_{12} & x_{22} & x_{32} & x_{42} \\ x_{13} & x_{23} & x_{33} & x_{43} \end{bmatrix}$$

```
Cmd> X
(1,1)        6        2        8
(2,1)        8        5        6
(3,1)        1        2        3
(4,1)        9        4        5

Cmd> X'
(1,1)        6        8        1        9
(2,1)        2        5        2        4
(3,1)        8        6        3        5

Cmd> X''
(1,1)        6        2        8
(2,1)        8        5        6
(3,1)        1        2        3
(4,1)        9        4        5
```

Note that $\mathbf{X}'' = \mathbf{X}$.

Standard Matrix Manipulation

Scalar Multiplication For a *scalar* c (a single number)

$$c\mathbf{X} = \mathbf{X}c = \begin{bmatrix} cx_{11} & cx_{12} & cx_{13} \\ cx_{21} & cx_{22} & cx_{23} \\ cx_{31} & cx_{32} & cx_{33} \\ cx_{41} & cx_{42} & cx_{43} \end{bmatrix}$$

```
Cmd> 3*X
(1,1)       18        6       24
(2,1)       24       15       18
(3,1)        3        6        9
(4,1)       27       12       15
```

Negative of a matrix

$$-\mathbf{X} = \begin{bmatrix} -x_{11} & -x_{12} & -x_{13} \\ -x_{21} & -x_{22} & -x_{23} \\ -x_{31} & -x_{32} & -x_{33} \\ -x_{41} & -x_{42} & -x_{43} \end{bmatrix}$$

```
Cmd> -X
(1,1)       -6       -2       -8
(2,1)       -8       -5       -6
(3,1)       -1       -2       -3
(4,1)       -9       -4       -5
```

Just scalar multiplication with $c = -1$.

Matrix Addition simply adds matching entries in the matrix one-by-one. Note: the matrices you add must have the same dimensions.

$$\begin{bmatrix} 6 & 2 & 8 \\ 8 & 5 & 6 \\ 1 & 2 & 3 \\ 9 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 2 \\ 7 & 8 & 9 \\ 7 & 3 & 4 \\ 3 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 10 \\ 15 & 13 & 15 \\ 8 & 5 & 7 \\ 12 & 4 & 13 \end{bmatrix}$$

```
Cmd> y <- enter(2 7 7 3 5 8 3 0 2 9 4 8)

Cmd> Y <- matrix(y,4);Y
(1,1)         2         5         2
(2,1)         7         8         9
(3,1)         7         3         4
(4,1)         3         0         8

Cmd> X+Y
(1,1)         8         7        10
(2,1)        15        13        15
(3,1)         8         5         7
(4,1)        12         4        13
```

Matrix Subtraction simply subtracts matching entries in the matrix one-by-one. Note: the matrices you add must have the same dimensions.

$$\begin{bmatrix} 6 & 2 & 8 \\ 8 & 5 & 6 \\ 1 & 2 & 3 \\ 9 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 5 & 2 \\ 7 & 8 & 9 \\ 7 & 3 & 4 \\ 3 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 1 & -3 & -3 \\ -6 & -1 & -1 \\ 6 & 4 & -3 \end{bmatrix}$$

```
Cmd> X-Y
(1,1)         4        -3         6
(2,1)         1        -3        -3
(3,1)        -6        -1        -1
(4,1)         6         4        -3

Cmd> X + X'
ERROR: Dimension mismatch for + near X + X'
```

Non-Standard Matrix Manipulation

These are things we can do with matrices in MacAnova (and perhaps elsewhere) that don't have standard matrix algebra counterparts.

Elementwise product. Note, this is *not* matrix multiplication.

$$\mathbf{X} * \mathbf{Y} = \begin{bmatrix} x_{11}y_{11} & x_{12}y_{12} & x_{13}y_{13} \\ x_{21}y_{21} & x_{22}y_{22} & x_{23}y_{23} \\ x_{31}y_{31} & x_{32}y_{32} & x_{33}y_{33} \\ x_{41}y_{41} & x_{42}y_{42} & x_{43}y_{43} \end{bmatrix}$$

```
Cmd> X*Y
(1,1)       12        10        16
(2,1)       56        40        54
(3,1)        7         6        12
(4,1)       27         0        40
```

Elementwise division

$$\mathbf{X}/\mathbf{Y} = \begin{bmatrix} x_{11}/y_{11} & x_{12}/y_{12} & x_{13}/y_{13} \\ x_{21}/y_{21} & x_{22}/y_{22} & x_{23}/y_{23} \\ x_{31}/y_{31} & x_{32}/y_{32} & x_{33}/y_{33} \\ x_{41}/y_{41} & x_{42}/y_{42} & x_{43}/y_{43} \end{bmatrix}$$

```
Cmd> X/Y
WARNING: Zero divide set to MISSING
(1,1)            3           0.4             4
(2,1)       1.1429         0.625       0.66667
(3,1)      0.14286       0.66667          0.75
(4,1)            3       MISSING         0.625
```

Elementwise powers

$$\mathbf{X}^c = \begin{bmatrix} x_{11}^c & x_{12}^c & x_{13}^c \\ x_{21}^c & x_{22}^c & x_{23}^c \\ x_{31}^c & x_{32}^c & x_{33}^c \\ x_{41}^c & x_{42}^c & x_{43}^c \end{bmatrix}$$

$$c^{\mathbf{X}} = \begin{bmatrix} c^{x_{11}} & c^{x_{12}} & c^{x_{13}} \\ c^{x_{21}} & c^{x_{22}} & c^{x_{23}} \\ c^{x_{31}} & c^{x_{32}} & c^{x_{33}} \\ c^{x_{41}} & c^{x_{42}} & c^{x_{43}} \end{bmatrix}$$

```
Cmd> X^2
(1,1)      36        4       64
(2,1)      64       25       36
(3,1)       1        4        9
(4,1)      81       16       25

Cmd> 2^X
(1,1)      64        4      256
(2,1)     256       32       64
(3,1)       2        4        8
(4,1)     512       16       32

Cmd> X^Y
(1,1)          36          32          64
(2,1)   2.0972e6    3.9062e5    1.0078e7
(3,1)           1           8          81
(4,1)         729           1    3.9062e5
```

Works as long as matrix dimensions match.
Add a constant to a matrix

$$\mathbf{X} + c = \begin{bmatrix} x_{11}+c & x_{12}+c & x_{13}+c \\ x_{21}+c & x_{22}+c & x_{23}+c \\ x_{31}+c & x_{32}+c & x_{33}+c \\ x_{41}+c & x_{42}+c & x_{43}+c \end{bmatrix}$$

```
Cmd> X+6
(1,1)        12         8        14
(2,1)        14        11        12
(3,1)         7         8         9
(4,1)        15        10        11
```

Add a (column) vector to a matrix

$$\mathbf{X} + v = \begin{bmatrix} x_{11}+v_1 & x_{12}+v_1 & x_{13}+v_1 \\ x_{21}+v_2 & x_{22}+v_2 & x_{23}+v_2 \\ x_{31}+v_3 & x_{32}+v_3 & x_{33}+v_3 \\ x_{41}+v_4 & x_{42}+v_4 & x_{43}+v_4 \end{bmatrix}$$

```
Cmd> v <- run(4);v
(1)         1         2         3         4

Cmd> v+X
(1,1)         7         3         9
(2,1)        10         7         8
(3,1)         4         5         6
(4,1)        13         8         9
```

Add a (row) vector to a matrix

$$\mathbf{X} + w = \begin{bmatrix} x_{11}+w_1 & x_{12}+w_2 & x_{13}+w_3 \\ x_{21}+w_1 & x_{22}+w_2 & x_{23}+w_3 \\ x_{31}+w_1 & x_{32}+w_2 & x_{33}+w_3 \\ x_{41}+w_1 & x_{42}+w_2 & x_{43}+w_3 \end{bmatrix}$$

```
Cmd> w <- rep(2,3)';w
(1,1)         2         2         2

Cmd> X+w
(1,1)         8         4        10
(2,1)        10         7         8
(3,1)         3         4         5
(4,1)        11         6         7
```

Operate on columns Sums, products, sorting, minima, maxima, etc.

```
Cmd> sum(X)
(1,1)        24        13        22

Cmd> prod(X)
(1,1)       432        80       720

Cmd> min(X)
(1,1)         1         2         3

Cmd> max(X)
(1,1)         9         5         8

Cmd> sort(X)
(1,1)         1         2         3
(2,1)         6         2         5
(3,1)         8         4         6
(4,1)         9         5         8

Cmd> sort(X,down:T)
(1,1)         9         5         8
(2,1)         8         4         6
(3,1)         6         2         5
(4,1)         1         2         3
```

Try something useful.

```
Cmd> sum(X)/4
(1,1)     6     3.25      5.5

Cmd> m <- sum(X)/4;m # col means
(1,1)     6     3.25      5.5

Cmd> D <- X-m;D # devs from means
(1,1)     0    -1.25      2.5
(2,1)     2     1.75      0.5
(3,1)    -5    -1.25     -2.5
(4,1)     3     0.75     -0.5

Cmd> sum(D^2)/3 # col vars
(1,1)    12.667      2.25    4.3333

Cmd> c1 <- X[,1];c1
(1,1)             6
(2,1)             8
(3,1)             1
```

```
(4,1)            9

Cmd> grade(c1)
(1,1)            3
(2,1)            1
(3,1)            2
(4,1)            4
```

`grade()` finds index of smallest, next smallest, etc.
Rearrange rows of `X` so that first column is in ascending order.

```
Cmd> X[grade(c1),]
(1,1)            1            2            3
(2,1)            6            2            8
(3,1)            8            5            6
(4,1)            9            4            5
```