Fractional Factorials

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In many situations we can identify lots of potential treatment factors — 10, 15 maybe 20 factors.

The <u>smallest</u> complete 10 factor design is the 2^{10}; it has 1,024 treatments.

The <u>smallest</u> complete 20 factor design has 1,048,576 treatments.

These designs are simply too large to be feasible.

One good thing about factorials is that they allow us to estimate interactions.

One bad thing about factorials is that they “spend” degrees of freedom (which correspond to resources) estimating high order interactions.

For example, in a 2^8 design (255 treatment df), 163 of the 255 df are used estimating 4, 5, 6, 7, and 8 way interactions.

This is good if the interactions are important, but wasteful if we think high order interactions are small relative to main effects and low order interactions.

Elephant and the Flea Principle

When you want to weigh the elephant, don't worry about his fleas.

If you think main effects and low order interactions are big and high order interactions are small, then don't sweat the high order interactions.

(Of course, if high order interactions aren't small, you could have a problem.)

Fractioning

As an alternative to a full factorial, suppose that we keep all of the factors but only run part of the factorial design, a fraction of the factorial.

How do we choose the fraction?

How do we analyze the results?

We have less data, what did we lose going to a fraction?

Did we gain anything going to a fraction?

Fractional factorials are smaller designs that let us look at main effects and (potentially) low order interactions.

This makes sense in situations such as:

1. We believe many of the factors are inert and we are simply screening to find the factors that are not inert.
2. We believe that any high order interactions are small enough to ignore (elephant and flea principle).
3. We must shrink the design and are willing to take certain risks (discussed below) to do so.

Fractioning the two-series

We will discuss “regular” fractions of the two-series design. There are non-regular fractions, and you can fraction other designs besides the two-series.

We will build half fractions, quarter fractions, eighth fractions, and so on.

The shorthand is 2^{k-1} for a half fraction of a 2^k, 2^{k-2} for a quarter fraction, and so on. In general, 2^{k-p}.

A flip suggestion for generating a 2^{k-p} design would be to confound a 2^k into 2^p blocks, but only run one of the blocks.

In fact, that is exactly equivalent to what we do.

In practice we don't do it that way because there is an easier way to generate the same design.

Choose p defining contrasts. These will be generators for our design.

For one defining contrast W, we will use either the block with W=1 or the block with W= –1.

By convention, I is a column of all ones. Thus we will either have I=W or I=–W.

Consider a 2^{3-1} with I=ABC. This is the fraction where the ABC interaction is always 1.

	A	B	C	AB	...	ABC
a	1	−1	−1	−1		1
b	−1	1	−1	−1		1
c	−1	−1	1	1		1
abc	1	1	1	1		1

Note that the C contrast matches the AB contrast, and ABC is always 1. There are also always an odd number of A, B, and C at the high level in this particular design.

Try this again with I=–ABC. This is the fraction where the ABC interaction is always –1.

	A	B	C	AB	...	ABC
(1)	–1	–1	–1	1		–1
ab	1	1	–1	1		–1
ac	1	–1	1	–1		–1
bc	–1	1	1	–1		–1

Note that the C contrast is always the negative the AB contrast, and ABC is always –1. There are also always an even number of A, B, and C at the high level.

Note: you cannot count on even begin –1 and odd being 1. Those will change depending on the number of factors in the model and the number of factors in the defining contrast.

To get a quarter fraction, we need two defining contrasts W_1 and W_2. Our fraction will either be the –1/–1, –1/1, 1/–1, or 1/1 combinations.

Let's look at a 2^{5-2} generated by I=ACE and I=–BCD.

	A	B	C	D	E	AC	BC	...	ACE	BCD	ABDE
e	–1	–1	–1	–1	1	1	1		1	–1	–1
a	1	–1	–1	–1	–1	–1	1		1	–1	–1
bde	–1	1	–1	1	1	1	–1		1	–1	–1
abd	1	1	–1	1	–1	–1	–1		1	–1	–1
cd	–1	–1	1	1	–1	–1	–1		1	–1	–1
acde	1	–1	1	1	1	1	–1		1	–1	–1
bc	–1	1	1	–1	–1	–1	1		1	–1	–1
abce	1	1	1	–1	1	1	1		1	–1	–1

Note that ACE is always 1 and BCD is always –1; also, ABDE is always –1. Note further that the D contrast is always the reverse of the BC contrast, and the E contrast is the same as the AC contrast.

Aliasing

The D contrast is the reverse of the BC contrast, so D cannot be distinguished from –BC; they are aliases of each other.

The E contrast is the same as the AC contrast, so E cannot be distinguished from AC; they are aliases of each other.

In a half fraction, each factorial contrast will come with two names (aliases). In a quarter fraction, each factorial contrast will come with four names (aliases).

We did not get something for nothing!

> The price we pay for fractioning is that every factorial degree of freedom comes with multiple aliases. Within the context of the experiment, we <u>cannot</u> distinguish between the aliases.

However, if potentially large main effects (the elephants) are always aliased with assumed to be small interactions (the fleas), then we are essentially looking at the main effects (when weighing an elephant with its fleas, you're really getting a weight of the elephant).

Let's repeat.

We have I=–BCD, and we also have D=–BC.

We have I=ACE, and we also have E=AC.

We have I=–ABDE, and we also have AB=–DE (not shown above).

Finding aliases

In a quarter fraction, we have I$=W_1$ and I$=W_2$, so we also have I$=W_1W_2$.

In our example: I=ACE and I=–BCD, so we also have I=–ABDE (reduce exponents mod 2).

Again, we have I $= W_1 = W_2 = W_1W_2$, so we also have A = AI $= AW_1 = AW_2 = AW_1W_2$

I = ACE = –BCD = –ABDE, so
A = CE = –ABCD = –BDE (reduce exponents mod 2).

We can do the whole set of effects, again by reducing exponents mod 2:

$I = ACE = –BCD = –ABDE$
$A = CE = –ABCD = –BDE$
$B = ABCE = –CD = –ADE$
$C = AE = –BD = –ABCDE$
$D = ACDE = –BC = –ABE$
$E = AC = –BCDE = –ABD$
$AB = BCE = –ACD = –DE$
$ABC = BE = –AD = –CDE$

When you design a fraction, make sure than you do not alias important (non-ignorable) things together.

Generating fractions made easy

Now there is a welcome topic!

In a 2^{k-p} design, there is some set of k-p factors for which the fraction is a complete design (when you ignore the other p factors).

In fact, there are usually lots of sets of k-p factors that will work. You just need a set where none of the effects or interactions of those factors is aliased to I.

In our I = ACE = –BCD = –ABDE example, any set of three <u>except</u> ACE and BCD will work.

I call this included factorial the base factorial.

It will be the case that any other factor will be aliased to some interaction of the factors in the base factorial.

In our I = ACE = –BCD = –ABDE example, A, B, and C can form a base factorial. Then D=–BC and E=AC.

Or we could have used A, D, and E for our base factorial. Then B=–ADE and C=AE.

What we do is build the base factorial, and then tack on the extra factor levels.

base effects	A	B	C	D=–BC	E=AC	final effects
(1)	–1	–1	–1	–1	1	e
a	1	–1	–1	–1	–1	a
b	–1	1	–1	1	1	bde
ab	1	1	–1	1	–1	abd
c	–1	–1	1	1	–1	cd
ac	1	–1	1	1	1	acde
bc	–1	1	1	–1	–1	bc
abc	1	1	1	–1	1	abcd

Analysis

Analyze a 2^{k-p} as a complete factorial in the base factors.

But don't forget the aliasing!

Suppose you analyze our example using factors A, B, and C, and the BC interaction alone looks big.

Remember that BC=ABE=–D=–ACDE. What you are seeing is probably, but not for certain, the D main effect.

Portents of things to come. We are still in our example with
$I = ACE = -BCD = -ABDE$

Suppose that
$A = CE = -ABCD = -BDE$
$C = AE = -BD = -ABCDE$
$E = AC = -BCDE = -ABD$
look big.

Is that three main effects, or is it A, C, and AC (or A, E and AE) (or C, E, and CE)?

You cannot distinguish these four situations from within the data. There is a potential price for using a fraction!

Resolution and aberration

In a resolution R design, no j-factor effect is aliased to anything with fewer than R-j factors.

In our regular designs, resolution is just the minimum number of letters in an alias of I.

Resolution is usually denoted by a Roman numeral.

Our $I = ACE = -BCD = -ABDE$ example is a 2^{5-2}_{III} design.

Higher resolution is better.

Resolution III designs don't alias main effects, but they do alias main effects and two factor interactions. Resolutions III designs are sometimes called "main effects" designs.

Resolution IV designs don't alias main effects to two factor interactions, but they do alias two factor interactions to other two factor interactions.

Resolution V designs do not alias any main effects or two factor interactions to each other.

Not only do we want the resolution to be as high as possible, we want as few as possible of those short aliases.

Aberration is really just a count of the number of alias of I of different lengths (3, 4, 5, etc letters).

Design A has smaller aberration than design B if the A has a smaller number in the first digit where they differ.

Lower aberration is better.

Three different 2_{IV}^{7-2} plans. Each design has lower aberration than the ones above it.

	Aberration				
Design	3	4	5	6	7
I=ABCF=BCDG=ADFG	0	3	0	0	0
I=ABCF=ADEG=BCDEFG	0	2	0	1	0
I=ABCDEF=ABCEG=DEFG	0	1	2	0	0

Projection

Any resolution R design contains a complete factorial in any R-1 factors.

This factorial could be replicated.

There could be sets of R or more factors that also form a complete factorial, but no guarantees.

If you think that there shouldn't be more than 3 active factors (with the rest inert), then a resolution IV design would allow you to get all their main effects and interactions, and you don't even need to know which three.

The trick is the assumption that there can't be more than three.

One possible approach to analyzing a fraction is to do the Daniel plot to identify a few active factors, then project down onto those factors.

Projecting down means just looking at those factors (and their interactions), using other df for error.

This is common practice, but still somewhat suspect in that you have made the big factors "treatments" and the little factors "error". The p-values are not unassailable.

Blocking (confounding)

We know how to block a two-series into 2, 4, 8, etc blocks.

We do exactly the same thing in a fraction, but each confounded effect has multiple aliases.

Be sure not to confound something important.

Our example has I $=$ ACE $=$ –BCD $=$ –ABDE, and one of the alias sets is AB $=$ BCE $=$ –ACD $=$ –DE.

To get two blocks of four, put the four runs with an even number of a's and b's in one block and the odds in the other.

The treatments are e, a, bde, abd, cd, acde, bc, and abcd. The two blocks are thus (e, abd, cd, abcd) and (a, bde, acde, bc).

Realistically, doing this would try to get too much out of 8 runs (5 main effects and a block).

Dealiasing

We have seen that aliasing can sometimes leave us in a situation where there is more than one reasonable explanation of the "large" effects.

There is no way to resolve this ambiguity internal to the data; we can only resolve this with more information.

More information usually means collecting more data, but it could, in principle, mean using outside information (the literature, other experiments, etc) to resolve the question.

We will discuss the simplest form(s) of collecting more data to remove troublesome aliasing.[1]

The practice of "breaking" the bad aliasing is called dealiasing.

Our version of dealiasing is based on running additional fractions of the factorial.

For example, we might go from a one-eighth replication to a one-quarter replication. The question is, which additional eight fraction should we run?

[1] Sounds like removing those troublesome stains from your laundry.

Let's think about families of fractions.

For a half fraction of a 2^4 we could have two members of the family: I=ABCD and I=–ABCD.

For our sample 2^{5-2}, there are four members of the family:
I = ACE = BCD = ABDE
I = ACE = –BCD = –ABDE
I = –ACE = BCD = –ABDE
I = –ACE = –BCD = ABDE

A different set of generators give us a different family of fractions.

For example, here is a different 2^{5-2} with a different family:
$I = ABD = CDE = ABCE$
$I = ABD = -CDE = -ABCE$
$I = -ABD = CDE = -ABCE$
$I = -ABD = -CDE = ABCE$

When we have a fraction, we will dealias by choosing another fraction from the same family.

Suppose we have the fraction I = ABD = –CDE = –ABCE

Now we run another fraction, say I = –ABD = CDE = –ABCE

Any aliasing in common to the two fractions remains as aliasing in the combined design. In this case, I = –ABCE is aliased in the combined design.

Any aliasing that changes between the two fractions is confounded with fraction differences. As we really ran the fractions in two separate goes, this is really block differences. ABD = –CDE is confounded with the block differences.

Example. We run a 2^{7-4} design with generators ABCD, BCE, ACF, and ABG.

The aliases of I are $I = ABCD = BCE = ACF = ABG = ADE = BDF = ABEF = CDEF = CDG = ACEG = BDEG = BCFG = ADFG = EFG = ABCDEFG$.

If the contrasts corresponding to A, D, and F are big, we are OK, it's probably the main effects. No dealiasing is needed.

But if the contrasts corresponding to A, D, and E are big, then we have ambiguity, and we can fix it by dealiasing.

Any other member of the family of fractions that has I = –ADE will break that aliasing. For example,

I = ABCD = –BCE = –ACF = –ABG = –ADE = –BDF = ABEF = CDEF = –CDG = ACEG = BDEG = BCFG = ADFG = –EFG = –ABCDEFG.

Aliasing in common: I = ABCD = ABEF = CDEF = ACEG = BDEG = BCFG = ADFG

Aliasing that changed: –BCE = –ACF = –ABG = –ADE = –BDF = –CDG = –EFG = –ABCDEFG.

We began with resolution III and ended with resolution IV.

Resolution IV is the best you can do with a 2^{7-3}, so we were not much worse off doing two sixteenths than if we had done an eighth to begin with.

In some cases, you could get more resolution if you had designed the larger experiment from the start, but you miss the chance of finding all you need with a smaller design.

Sequences of small fractions can sometimes be better than one larger fraction.

Example. 7 factors, at most 32 runs. Assume that at most 4 factors are active, and assume that all three way interactions are negligible.

The best you can do with a 32 run design is 2^{7-2}_{IV}. For example, $I = ADEF = ABCDG = BCEFG$.

Resolution IV is not really good enough if ADEF happen to be your active factors, because you will still alias some potentially active two factor interactions together.

However, try starting with a 2^{7-3}_{IV}. For example, I $=$ BCDE $=$ ACDF $=$ ABCG $=$ ABEF $=$ ADEG $=$ BDFG $=$ CEFG.

This design is good enough for any set of four active factors except the seven sets of four that form aliases of I. Most of the time, we can get what we need with 16 units.

If our big effects are from one of those sets of four, say B, C, D, and E, then run another 16 runs that change the sign of BCDE. For example, I $=$ –BCDE $=$ ACDF $=$ ABCG $=$ –ABEF $=$ –ADEG $=$ BDFG $=$ –CEFG.

Combined with the first fraction this breaks the BCDE aliasing.

Take away points.

- Running 32 runs at once could not get the resolution we needed, although it is good enough for nearly all sets of four active factors.
- Often, 16 runs is good enough for sets of four active factors.
- Running 16 runs, taking a look, and then running another 16 runs does not get resolution V, but it can break all important aliasing.
- The trick is the first 16 runs scout out the ground so that we can choose the second 16 runs optimally.

This sort of thing works best when signal is large relative to noise.

Fold over

Fact 1: it is easy to generate resolution III designs, even huge ones.

Fact 2: it is easy to take a $2_{III}^{(k-1)-p}$ design and turn it into a 2_{IV}^{k-p} design. This process is called fold over.

Putting these together means that it is easy to generate resolution IV designs, even huge ones.

Construct a 2^{k-p}_{III} design. We can do this as long as $2^{k-p} > k$.

Simply begin with a base factorial in k-p factors, then alias the remaining p factors to any interactions of the base factors.

This will be a resolution III design.

Fold over.

1. Write out your $2_{III}^{(k-1)-p}$ design using $+/-$ notation.

2. Make another set of $2^{(k-1)-p}$ by <u>reversing all the signs</u> in the first $2_{III}^{(k-1)-p}$ design.

3. Add the kth factor by making it – for the first $2^{(k-1)-p}$ runs and $+$ for the last $2^{(k-1)-p}$ runs.

This process breaks all the odd length aliases, typically leaving us with resolution IV.

A	B	C	D=AB	E=AC	F=BC	G=ABC	H
$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$
$+$	$-$	$-$	$-$	$-$	$+$	$+$	$-$
$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$
$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$
$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$
$-$	$+$	$+$	$-$	$-$	$+$	$-$	$-$
$+$	$+$	$+$	$+$	$+$	$+$	$+$	$-$
$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$
$-$	$+$	$+$	$+$	$+$	$-$	$-$	$+$
$+$	$-$	$+$	$+$	$-$	$+$	$-$	$+$
$-$	$-$	$+$	$-$	$+$	$+$	$+$	$+$
$+$	$+$	$-$	$-$	$+$	$+$	$-$	$+$
$-$	$+$	$-$	$+$	$-$	$+$	$+$	$+$
$+$	$-$	$-$	$+$	$+$	$-$	$+$	$+$
$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$

What happened to the generators?

D=AB becomes D=–ABH
E=AC becomes E=–ACH
F=BC becomes F=–BCH
G=ABC remains G=ABC

That one wasn't so bad, and it could be in a table.

What if you want a 2^{21-15}_{IV}; you're not going to find that one in the back of the book.

Start with a 2^{20-15}_{III}, then fold over.