Convenient Interactive Computing for Coherent Imprecise Probability Assessment*

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Abstract

A generalization of deFinetti's Fundamental Theorem of Probability facilitates coherent assessment, by iterated natural extension, of imprecise probabilities or expectations, conditional and unconditional. Point values are generalized to assessed bounds, accepted under weak coherence, that is, allowing the input of redundant, or slack, bounds. The method is realized in a convenient interactive computer program, demonstrated in a tutorial here, and made available as open source code. This suggests that the fees of consulting experts not be paid unless their reported probabilities cohere.

Keywords: Assessment; Imprecise Probabilities; Previsions; Coherent probabilities; Natural Extension; Interactive Computing.

1 Introduction

Back when my father was living, I had a serious conversation with his physician, an experience which continues to contribute to my interest in coherent probability. I was told that my father's chances of surviving his medical condition, event S, depended strongly on whether he had a systemic fungal infection, event F. The physician first informed me that $P(S)$ equaled 60%. Then when I asked about the conditional probabilities, he said that $P(S|F)$ was 20% and $P(S|F')$ lay between 70% and 80%, where $F' =$ (not F). Finally, when I asked him the chances of a fungal infection, he replied that $P(F)$ was 60%. A calculation by the Law of Total Probability, however, yields the interval, $.40 \leq P(S) \leq .44$, which excludes the reported value $P(S) = .60$.

(I did not discuss this anomally with the physician, in part, because of an earlier experience when I had asked my wife's allergist about the probability of my wife succumbing to a bee sting. The allergist had replied, "You can't think of it in terms of probabilities." citing as a reason, "She might die !" I had gained no attention on that occasion by suggesting she also might die in an automobile accident traveling to obtain her weekly injection.)

Coherence would say that the given values for $P(S|F), P(S|F')$, and $P(F)$ require that $P(S)$ lie between .40 and .44. Interpreting probabilities as personal fair betting prices, this says that these given values, together with the non-coherent value $P(S) = .60$, can be used to construct a sure-loss combined bet, as follows. Defining events as $0, 1$ random quantities, consider the combination of bets: a bet for $\$S$; two bets, against $\$.20F$ and $\$.80F'$, respectively; and two called-off bets, against $\$S$ given F, and against $\$S$ given F'. This would have the combined net gain,

$$\begin{aligned} \$(S - .60) - \$[(S - .20)F + .20(F - .60)] \\ -\$[(S - .80)F' + .80(F' - .40)] \\ = -\$.16 \end{aligned} \tag{1}$$

The random quantities here cancel, and this reduces to a sure-loss of $.16 .

My former colleague, David Lane assures me that, when medical doctors use probabilities, they are referring to relative frequencies in a patient population. They are giving personal estimates of population frequencies, rather than using probabilities as direct expressions of personal uncertainty about a patient. Non-coherence in this context means that there could not exist a population (finite or infinite) having such frequencies. So, yet again, coherence would be required for rational discussion, perhaps even for rational thought.

2 Coherence

An event being the special case of a random quantity with range-set {0, 1}, a probability is an expected value. The term "prevision" can serve for both, so we present coherent assessment methods, useful both for probabilities of events and expectations of random quantities, referred to, in general, as "previsions" $P(X)$ of random quantities X.

By "coherent" is meant that a set of assessed previsions, $P(X_1), ..., P(X_n)$ agree with at least one full probability distribution satisfying the probability axioms. Logical relations between events, or more generally, mathematical relations between random quantities, impose linear coherence constraints on their previsions. These are easy to violate inadvertantly, since the constraints are not easily known without constructing the partition consisting of all the constituent events, $C_1, ..., C_N$, of the given random quantities, $X_1, ..., X_n$, that is, all the possible and-combinations, $C_j = (B_1$ and ... and $B_n)$, where each B_i is a value-event, $(X_i = x_i)$, for some value x_i possible for the random quantity X_i. Since not all such and-combinations are possible, $N \leq r_1...r_n$, where each r_i is the number of possible values x_i of X_i.

If $\boldsymbol{X} = (X_1, ..., X_n)^T$ is the vector of events under discussion and $\boldsymbol{C} = (C_1, ..., C_N)^T$ is the vector of their constituent events, then $\boldsymbol{X}$ and $\boldsymbol{C}$ are related by the linear system,

$$\boldsymbol{X} = R\,\boldsymbol{C}, \tag{2}$$

where R is the joint-range matrix of the X_i's, that is, the matrix of columns consisting of all the possible outcome vectors for $\boldsymbol{X}$. In the case of events X_i, the matrix R is their truth table, coded as 1 for True, 0 for False. Then coherence is equivalent to the requirement that the previsions $\boldsymbol{p} = P(\boldsymbol{X})$ satisfy the system,

$$\boldsymbol{p} = R\ \boldsymbol{q}, \tag{3}$$

for some vector $\boldsymbol{q}$, where

$$\boldsymbol{q} \geq 0 \text{ (nonneg. coords.), and } q_1 + ... + q_N = 1. \tag{4}$$

One can interpret each coordinate q_j as the (unspecified) probability of the corresponding constituent event C_j, $P(Cj) = q_j$.

Equations (3), (4) can be understood geometrically to say that previsions $\boldsymbol{p} = P(\boldsymbol{X})$ cohere if and only if they lie in the convex hull of the columns of the joint-range matrix R. Finally, if prevision values (or mere bounds) are imposed on P($\boldsymbol{X}$) (or linear combinations thereof), then further linear restrictions are implied, and these reduce the convex hull to a convex subset, consisting of all the available coherent probabilities.

Fortunately, a generalization of de Finetti's Fundamental Theorem of Probability [18, 19, 3] provides a practical method to assess values and/or bounds for the previsions of a list of random quantities. Such prevision restrictions can be assessed sequentially, checking at each nth step to solve, by linear programming, for the interval of available cohering values, $p_* \leq P(X_n) \leq p^*$, before choosing the value or bounds for the next prevision $P(X_n)$. The computed interval $[p_*, p^*]$ was called, by Walley [22] the "natural extension" of the previsions and bounds that are so-far assessed. Thus, one assesses previsions by iterating an extend-assess cycle, as follows:

1. (**Extend Step.**) Given values (or mere bounds) for $n-1$ previsions $P(X_i) = p_i$, $i = 1, ..., n-1$, coherently extend consideration to $P(X_n)$. Coherence requires that $P(X_n) = p_n$, for some p_n, where p_n lies in the extension interval,

   $$p_* \leq p_n \leq p^*, \tag{5}$$

   where the endpoints p_*, p^* can be obtained by linear programming with the linear restrictions (3), (4) and the objective function $P(X_n)$ of the variables $\boldsymbol{q} = P(\boldsymbol{C})$. If the input previsions $p_1, ..., p_{n-1}$ (or bounds) are not coherent, the linear-programming calculation will report "no feasible solution".

2. (**Assess Step.**) Choose a value $P(X_n) = p_n$ to satisfy Equation (5), with regard to the calculated extension bounds p_*, p^*. Alternatively, choose mere bound(s), a_n, b_n, $a_n \leq P(X_n) \leq b_n$, that do not violate the extension bounds.

A value (or mere bounds) for a conditional prevision $P(X|Y) = p$, where Y denotes an event, can be input as a linear constraint(s) in an extension calculation, $P(XY) - p\ P(Y) = 0$. A conditional prevision can also play the role of the objective function in such a calculation. Optimizing the ratio, $P(X|Y) = P(XY)/P(Y)$, would constitute a fractional-programming problem, which by a simple change-of-variables, can be recast as an equivalent linear-programming problem.

The general method allows the assessment of mere bounds, or intervals, in place of point values for previsions and conditional previsions, as in $a_n \leq P(X_n) \leq b_n$. The linear programming can take this generalization in its stride, but what about the concept of coherence? What does it mean for prevision bounds to be coherent or not coherent? We use a weaker notion of coherence for inequalities than Walley [22], who disallows extraneous slack bounds. Weak coherence has been termed "g-coherence" by Biazzo and Gilio [3].

If mere bounds are input for $P(X_1), ..., P(X_{n-1})$, instead of precise values, the output extension interval consists of all the available values p_n for the further prevision $P(X_n)$ for which there exists at least one mutually coherent list of precise values, $p_1, ..., p_n$, with the first $n-1$ values, $p_1, ..., p_{n-1}$, satisfying the input bounds. And, of course, for each such initial list of $n-1$ precise values, the corresponding cohering values for the further prevision $P(X_n)$ would form a subinterval of the output interval. This was defined as the problem of probability logic by Hailperin [15] (following Boole [2]), included as "natural extension" by Walley [22], and presented in a generalization of deFinetti's Fundamental Theorem by Lad, Dickey, and Rahman [18, 19]. The latter two papers are the basis for the algorithm coded in the present program. A prototype program, written in Mathematica in 1991, has had limited distribution. See also Biazzo and Gilio [3].

So, what assessed further bounds should one say "cohere" with the output extension interval?

Definition (Weak Coherence). Assessed bounds that do not contradict the output bounds will be said to cohere weakly with the given input bounds. An assessed lower (upper) bound must not lie above (below) the output upper (lower) bound, that is, the assessed interval must overlap the extension interval. Also, of course, an assessed lower (upper) bound must not be higher (lower) than the corresponding assessed upper (lower) bound. Such weak coherence is directly equivalent to the prevention of sure-loss combined bets.

3 A Tutorial

COHERE is an interactive computer program to enable the assessment of coherent probabilities of events and/or expected values of random quantities. In its central function, the program determines what some coherent previsions or prevision bounds imply about another prevision, how the previsions coherently "extend". This is useful as a generic step in the progressive assessment of a coherent set of previsions. The main idea is to enable the assessment of a list of mutuallly coherent previsions or prevision bounds, by successively choosing a value or bounds agreeing with a current extension interval. A related use might be to focus on the assessment of probability for a particular event, such as

MENU			
FILE:		ACTION:	
n	New	a	Assess
o	Open	au	Undo Assessments
s	Save	e	Extend
p	Print	eu	Undo Extension
q	Quit	t	Option

Figure 1: Program menu.

overall system failure for a complex engineering system. There could be many component events that have a bearing on the event of interest, the probabilities of which could be assessed, estimated, or bounded coherently.

The program COHERE is available under a GNU open source license as the files, cohere, lp_solve, at

$$\text{http://www.stat.umn.edu/\~{}dickey/public\_html/COHdistribute/} \tag{6}$$

This tutorial, coheretut.pdf, and its example input-output file, tutfile.coh, are also available at the same address.

This tutorial is built around the relatively simple example of diagnosing a disease (tuberculosis) from an adverse outcome of a screening test (positive skin test). Denote by T the event that a particular person, the patient, has the disease, nT the logical negation of T, and S the event that the patient has a positive skin test. We demonstrate use of the program to bound, and then to assess, the unconditional probability of S, and to bound the conditional probability of T given S. This is approached, first, by working with assessed values/bounds on the unconditional probability of T, and the conditional probabilities of S given T, and S given nT.

To start the program, enter the following command at a unix prompt, two pathnames, separated by a space:

$$\text{PATHPART1/cohere PATHPART2/tutfile.coh} \tag{7}$$

(In the School of Statistics, University of Minnesota, PATHPART1 and PATHPART2 are both stat.umn.edu/˜dickey/COHERE.) This command will first construct (if missing) a directory, COHuser/, named from your username, user. It will serve as your temporary working directory during program execution and it will be located within your original working directory. Following program execution, your original working directory will be restored automatically to its role as current working directory.

You will be confronted by the program menu (Fig. 1). Choose "a" in the menu (for "Assess") to view and edit the input-output file, iofile.coh. The program will respond by opening the vi editor, and displaying iofile.coh, as copied from tutfile.coh (see Fig 2). This will enable you to define events/quantities, to make assessments concerning previsions of these quantities, and to ask for the

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** TITLE/DESCR:        - SCREENING TEST FOR TB -
**  Separate fields by "; " (semicolon space(s)). Records (lines) by <Enter>.
** EVENT/QUANT DEFN FIELDS:  xname; descr; rangeSet(or"fun"); relation(or expr)
** PROB/EXPEC FIELDS:  (Indent)P(xname); bdL; bdU; "a"(Assess)or"e"(Extend)stepN

T;             Patient has TB;          (0, 1);        none
     P(T);          5.00e-05;      1.00e-04;      a

     P(T|S);        ;              ;              e


nT;            Doesn't have TB;         not $T;        none


S;             Pos skin test;           (0, 1);        none
     P(S|T);        eq;            1;             a

     P(S|nT);       1/20;          1/10;          a

     P(S);          ;              ;              e
```

Figure 2: Initial input file.

extension bounds on previsions of particular quantities, as implied by coherence. (By default, such questions will be asked, automatically, for those quantities for which you have assessed bounds, to reveal whether the assessed bounds are adding new information.)

Instead of the unix command with a filename argument (7), you can begin by entering just the command without arguments,"PATHPART1/cohere," and then use the program menu command "n", for "'new". Then the vi editor will display a new empty file, iofile.coh. Only the first four header lines will appear, with the first line wanting your title/description for the assessment task. When a star "$*$" is the first nonblank character in any line, the line is ignored by the program. Alternatively, the menu command "o", for "old", can be used to load a previous input-outout file.

Turning to Fig. 2, note the three left-justified lines that define the events T, nT, S. In each such line, there are four fields, separated by "; " (semicolon, space(s)). The field values, entered by the user, are:

1. The name of the event/quantity.

2. Any description.

3. The ordered range set of the event/quantity written as a Perl expression, e.g. "(0, 1)" for an event. Or, if the quantity is being defined as a function of preceding quantities, write a Perl expression for the function value,

e.g. "$\$A$ or $\$B$" for the event, $C = (A \text{ or } B)$. (A dollar sign signifies a variable in Perl.) More generally, you can enter a conditional range set. For example, if the current event C is implied by a previous event A, then the conditional range of C would be (1) if A, and $(0, 1)$ otherwise. This conditional range can be given by entering into field 3 the Perl expression, $\text{grep}((1 - \$A + \$\_), (0, 1))$.

4. Any relation to previously defined events/quantities, written as a boolean Perl expression ("none" or "no rel" for no relation). Another way to define a quantity in terms of previous quantities is to enter its (marginal) range set in field 3, e.g "(0, 1)", and just enter the defining relation in field 4, "$\$C == (\$A \text{ or } \$B)$" (double equal sign for the relation). A relation can appear in either field; e.g., the relation, A implies C, can be written in field 4 as "`$A <= $C`", or as "not ($\$A$ and (not $\$C$))". Expressions for multiple relations can be entered in field 4 by combining single relations by multiplication or logical conjunction ("$*$" or "and"). For example, "$(\$X3 \leq \$X2) * (\$X2 \leq \$X1)$" declares that $X3$ implies $X2$, and $X2$ implies $X1$.

The second and third types of line concern the previsions of quantities. The corresponding fields are:

1. P(X), where X is the name of an event/quantity. (An empty field is OK for the same name as the immediately preceding prevision line.)

2. Lower bound on prevision.

3. Upper bound on prevision.

4. Action code. Enter "a" for an assessment from the user, or "e" to request an extension by the program. (A step number will be appended automatically by the program.)

For a single point value, say .37, to be assessed, instead of bounds, enter "eq; ," ".37; " into the second and third fields, respectively. By default, the program will continue to calculate, at all subsequent steps, the extension intervals for each interval-assessed quantity and each quantity whose extension is requested. To override, and prevent the perpetual calculation of an extension interval, use "a!" or "e!", The step number will be incremented by the program whenever there is an extend action based on an additional assessment line.

To compute the extension bounds called for in iofile.coh (Fig. 2), first record in the editor any changes to the file (":w" in the control mode of vi). (In vi, the ESC key switches to control mode, whereas, "i" switches to insert mode.) You can record your changes by writing to iofile.coh when leaving vi (":wq"). Then choose the extend action in the program menu ("e"). After computing the extension, the program will return you to its menu, where you can view the output by again choosing the assess action ("a"). A display of the program-modified iofile.coh is given in Fig. 3.

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** PROB/EXPEC FIELDS:  (Indent)P(xname); bdL; bdU; "a"(Assess)or"e"(Extend)stepN

T;             Patient has TB;          (0, 1);        none
     P(T);          5.00e-05;      1.00e-04;      a1
     ;              [5e-05;        0.0001];       e1

     P(T|S);        [0.0004998;    0.001996];     e1


nT;            Doesn't have TB;         not $T;        none


S;             Pos skin test;           (0, 1);        none
T_S;           T and S;                 $T*$S;         none
S_nT;          S and nT;                $S*$nT;        none
     P(S|T);        eq;            1;             a1

     P(S|nT);       1/20;          1/10;          a1
     ;              [0.05;         0.1];          e1

     P(S);          [0.05005;      0.1001];       e1
```

Figure 3: First output.

Fig. 3 shows the computed extension bounds, as requested in Fig. 2. There are: (i) computed checks on the assessed bounds for $P(T)$ and $P(S|nT)$; (ii) implied bounds for the marginal probability of a positive skin test $P(S)$; and (iii) the implied bounds for the probability of tuberculosis conditional on a positive skin test $P(T\,|\,S)$. Note the new definition lines for the and-events $(S\,and\,T)$ and $(S\,and\,nT)$, automatically defined by the program to enable program use of the input information on the conditional probabilities, $P(S\,|\,T)$, $P(S\,|\,nT)$, and to enable calculation of the extension bounds on $P(T\,|\,S)$.

Finally, we can use the computed extension interval of the marginal probability of a positive skin test, $0.05005 \leq P(S) \leq 0.1001$, as a coherent guide for a further, precise assessment, $P(S) = 0.07$. It could be known, for example, that the relevant experimental frequency of positive test results is equal to this value. After input of this assessed value and the choice of extension "e" in the menu, the program outputs the corresponding step-2 extension intervals, as given in Fig. 4. Note that the interval for the conditional false-positive probability $P(S\,|\,nT)$ has now shrunk almost to a single value, and the "posterior" probability of the patient having the disease is now confined to the narrower interval, $0.0007143 \leq P(T\,|\,S) \leq 0.001429$.

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** PROB/EXPEC FIELDS:  (Indent)P(xname); bdL; bdU; "a"(Assess)or"e"(Extend)stepN

T;             Patient has TB;          (0, 1);        none
     P(T);          5.00e-05;      1.00e-04;      a1
     ;              [5e-05;        0.0001];       e1
     ;              [5e-05;        0.0001];       e2

     P(T|S);        [0.0004998;    0.001996];     e1
     ;              [0.0007143;    0.001429];     e2


nT;            Doesn't have TB;         not $T;        none


S;             Pos skin test;           (0, 1);        none
T_S;           T and S;                 $T*$S;         none
S_nT;          S and nT;                $S*$nT;        none
     P(S|T);        eq;            1;             a1

     P(S|nT);       1/20;          1/10;          a1
     ;              [0.05;         0.1];          e1
     ;              [0.06991;      0.06995];      e2

     P(S);          [0.05005;      0.1001];       e1
     ;              eq;            .07;           a2
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Figure 4: Second output.

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