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RESEARCH ARTICLE

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Production, interest, and saving in deterministic economies with additive endowments

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Abstract Stationary equilibria are constructed for a series of nonstochastic production economies in which the decisions of producers, wage earners, shareholders, and savers modulate, via a "production function", the endowment variables in an additive manner. The efficiency of each model is compared to that of a single agent who produces for personal consumption.

1 Introduction

The problem of determining optimal consumption and production/investment decisions, for a single economic agent and under uncertainty, has a long and venerable history. Its quantitative analysis goes back at least to Ramsey (1928); somewhat more recent works are those of Phelps (1962), Levhari and Srinivasan (1969),

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Samuelson (1969, 1971), Hakansson (1970), Scheinkman and Schechtman (1983), Ljungquist and Sargent (2000) and Bobenrieth et al. (2002). In most of these models the agent's production decision modulates (via the so-called "production function") a random endowment variable in a *multiplicative* fashion. For utility functions of the logarithmic- or power-type, one is then able to determine optimal consumption and production/investment rules in fairly explicit form, with infinitely many time-periods and discounting.

In this paper we broach the study of a model in which the agents' production decisions modulate (again via a "production function") the endowment variable in an *additive* fashion. We concentrate here on the case where the endowment variable is a known (deterministic) positive constant, the same from period to period. The advantage of this simplification is that it leads to exact solutions for very general production and utility functions, in several quite different models:

- A "Robinson Crusoe" economy, with a single agent who produces for his own personal consumption.
- A market economy with a continuum of identical agents, who hold cash and produce and/or purchase goods in a "cash-in-advance" fashion.
- A "cash-in-advance" market economy as above, but in which agents can also receive partial credit from a clearing house.
- A "cash-in-advance" market economy as above, but in which agents are now allowed to borrow from, or deposit cash into, a central bank that charges and pays interest at a fixed rate.
- A "cash-in-advance" market economy with a continuum of firms that produce goods, a continuum of agents who own the firms and purchase these goods for consumption, and a continuum of agents who subsist entirely on their savings and purchase goods for consumption.

With the assumption of certainty it is possible to provide exact solutions to all these models, for very general utility and production functions. These solutions can then be used to compare the efficiency of the various models.

In all the models except the last one, we assume the agents are independent producers with a production function $f(\cdot)$, and with a utility function $u(\cdot)$ for consumption. Both these functions are assumed to be concave, continuously differentiable, and increasing on $[0, \infty)$ with $u(0) = f(0) = 0$. We also assume that

$$\lim_{i \to \infty} f'(i) = 0.$$

In addition to the goods produced by the agents, we assume that every agent receives an endowment $y \geq 0$ of goods, which is constant and the same from period to period.

In a companion paper Geanakoplos et al. (2006) we extend these results to the case of real uncertainty for the endowment variable y, in the context of the models 1 and 4 of the present work.

2 Model 1: Robinson Crusoe

The simplest model is that of a single agent who produces for his personal consumption. The agent begins with $q \geq 0$ units of the good, puts i units into production with $0 \le i \le q$, and consumes the remaining $x = q - i$ units. The agent begins the next period with $f(i) + y$ units of the good, and the game continues. Let $V(q)$ be the supremum over all strategies of the expression

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n),$$

where x_n is the amount of the good consumed in the n^{th} period and $\beta \in (0, 1)$ is a given discount factor. Then the function $V(\cdot)$ satisfies the Bellman equation

$$V(q) = \sup_{0 \le i \le q} [u(q - i) + \beta V(f(i) + y)].$$

Theorem 1 *There are two cases.*

1. *If $f'(0+) < 1/\beta$, an equilibrium for the agent, at the initial state $q_1 = y$, is to hold y units of the good in each period and consume all of it.* (*There is no production in this case.*)
2. *If $f'(0+) \ge 1/\beta$, then there exists a number $i_1 \in [0, \infty)$ such that $f'(i_1) = 1/\beta$. An equilibrium for the agent, at the initial state $q_1 = f(i_1) + y$, is to hold q_1 units of the good in each period, to put i_1 units into production, and to consume $f(i_1) + y - i_1$ units.*

The proof of the theorem is in the Appendix.

Let $q_1 = f(i_1) + y$. Then in case 2 of the theorem, the agent's total return is

$$u(q_1 - i_1) + \beta u(q_1 - i_1) + \beta^2 u(q_1 - i_1) + \cdots = \frac{u(q_1 - i_1)}{1 - \beta}.$$

3 Model 2: a market for goods with cash-in-advance

In all the remaining models we assume that there is a continuum $I = [0, 1]$ of agents and denote a typical agent by $\alpha \in I$.

Each agent α holds cash $m_n^\alpha \ge 0$ and goods $q_n^\alpha \ge 0$ in every period $n = 1, 2, \ldots$. The goods are offered for sale in a market, and each agent α bids an amount of cash $b_n^\alpha \in [0, m_n^\alpha]$ for consumption.[1]

The price p_n for that period is formed as the ratio of the total bid, divided by the total amount of goods offered for sale. That is,

$$p_n = \frac{B_n}{Q_n},$$

where

$$B_n = \int_0^1 b_n^\alpha \, d\alpha, \quad Q_n = \int_0^1 q_n^\alpha \, d\alpha.$$

[1] Imagine that the good produced is milk; the agent has an agreement to sell all of his daily production to a secondary market or to a coöperative, but still has to go to the supermarket and buy a carton of pasteurized milk for his family's daily needs.

The agent's bid now buys him a quantity b_n^α / p_n of goods; after receiving these goods the agent puts a quantity $i_n^\alpha \in [0, b_n^\alpha / p_n]$ into production, and consumes the remaining amount

$$x_n^\alpha = \frac{b_n^\alpha}{p_n} - i_n^\alpha$$

of goods. Thus, the agent begins the next period with cash

$$m_{n+1}^\alpha = m_n^\alpha - b_n^\alpha + p_n q_n^\alpha \tag{1}$$

and goods

$$q_{n+1}^\alpha = f(i_n^\alpha) + y. \tag{2}$$

The game continues in the same way, with the total payoff to agent α being

$$\sum_{n=1}^{\infty} \beta^{n-1} u\left(x_n^\alpha\right).$$

Let M_n be the total amount of money held by all agents in period n; that is,

$$M_n = \int_0^1 m_n^\alpha \, \mathrm{d}\alpha.$$

Integrating with respect to α in equality (1) we get

$$M_{n+1} = M_n - B_n + \frac{B_n}{Q_n} . \, Q_n = M_n,$$

so the quantity of money is conserved. Let $M_1 = m$ be this common value.

Suppose that in equilibrium the prices p_n remain equal to a constant p. Then each agent faces a dynamic programming problem, whose optimal reward function $V(m, q)$ satisfies the Bellman equation

$$V(m, q) = \sup_{\substack{0 \le b \le m \\ 0 \le i \le b/p}} \left[u\left(\frac{b}{p} - i\right) + \beta \, V\big(m - b + pq, \, f(i) + y\big) \right].$$

Theorem 2 *There are two cases.*

1. *Suppose $f'(0+) < 1/\beta^2$. There is then an equilibrium with no production, in which every agent holds cash $M = m$ and goods $q = y$ in every period. Agents bid the maximum amount allowed $b = m$ and put $i = 0$ into production.*
2. *Suppose $f'(0+) \ge 1/\beta^2$. There exists then a number $i_2 \in [0, \infty)$ such that $f'(i_2) = 1/\beta^2$. There is an equilibrium in which each agent holds cash $M = m$ and goods $q = f(i_2) + y$ in every period. Agents bid the maximum amount $b = m$ allowed and put $i = i_2$ into production.*

In case 2 of the theorem, agents consume the amount

$$x_2 = f(i_2) + y - i_2$$

of goods in every period, whereas the corresponding quantity for Robinson Crusoe in model 1 is

$$x_1 = f(i_1) + y - i_1.$$

Now

$$i_1 = (f')^{-1}\left(\frac{1}{\beta}\right) > (f')^{-1}\left(\frac{1}{\beta^2}\right) = i_2,$$

and easy calculus shows that $\xi \mapsto f(\xi) - \xi$ is increasing on (i_2, i_1), so

$$x_1 > x_2$$

as well. In other words, the isolated Robinson Crusoe does a bit better (in terms of both production and consumption) than the agent in a cash-in-advance market economy.

The proof of Theorem 2 is omitted, as it is similar to the proofs given in the Appendix.

4 Model 3: a market with credit from a clearing house

In this model each agent α holds cash m_n^α and goods q_n^α in every period n, just as in model 2. The difference is that a clearing house is now willing to grant the agent *credit* for the goods q_n^α, so that he can make a larger bid.

Suppose first that the clearing house grants full credit $p_n q_n^\alpha$ to agent α, so that he is permitted to bid any amount

$$b_n^\alpha \in \left[0, m_n^\alpha + p_n q_n^\alpha\right].$$

There are two difficulties with this rule. First, the clearing house must predict the price before it has been formed as the ratio B_n/Q_n of total bids to goods. The second problem arises when all the agents make the maximum bid

$$b_n^\alpha = m_n^\alpha + p_n q_n^\alpha.$$

Integrate out α and divide by Q_n to see that

$$p_n = \frac{M_n}{Q_n} + p_n.$$

This can only be true if $M_n = 0$; that is, there must be no money in the economy.

To avoid this second difficulty, we shall assume that the clearing house grants only *partial credit*, by allowing bids

$$b_n^a \in \left[0, m_n^\alpha + \theta p_n q_n^\alpha\right],$$

where θ is a parameter with $0 \leq \theta < 1$. Except for this change, the description of this model is the same as that of model 2. In particular, the formulae (1) and (2) for the dynamics are the same; and the money supply $M_1 = m$ is conserved.

Theorem 3 *There are two cases.*

1. *Suppose* $f'(0+) < 1/(\beta\theta + \beta^2(1-\theta))$. *Then there is an equilibrium without production for which, in every period, each agent holds cash* $M = m$ *and goods* $q = y$*; bids the maximum amount* $b = m + \theta p y$ *allowed; and puts* $i = 0$ *into production. The equilibrium price is* $p = m/(y(1-\theta))$.
2. *Suppose* $f'(0+) \geq 1/(\beta\theta + \beta^2(1-\theta))$, *so that there exists a number* $i_3 \geq 0$ *with* $f'(i_3) = 1/(\beta\theta + \beta^2(1-\theta))$. *Then there is an equilibrium for which, in every period, each agent holds cash* $M = m$ *and goods* $q = f(i_3) + y$*; bids the maximum amount* $b = m + \theta p q$ *allowed; and puts* $i = i_3$ *into production. The equilibrium price is* $p = m/(Q(1-\theta))$, *where* $Q = f(i_3) + y$.

In case 2 of this theorem, agents consume

$$x_3 = f(i_3) + y - i_3.$$

Now $i_1 > i_3 > i_2$, where i_1 and i_2 are from Theorems 1 and 2, respectively. Consequently, the amounts consumed each period in the three models satisfy the same inequalities:

$$x_1 > x_3 > x_2.$$

Here x_3 is a function of θ and approaches x_1 as $\theta \nearrow 1$. In other words, the introduction of credit allows agents to "approach" Robinson Crusoe's welfare.

The proof of Theorem 3 is similar to those given in the Appendix.

5 Model 4: a market with cash-in-advance and a central bank

In this model the typical agent α holds cash m_n^α and goods q_n^α just as in models 2 and 3.

The goods are offered for sale in a market, just as before. The new feature is that agents are now allowed to borrow from, or deposit cash into, a central bank that charges and pays interest at a fixed rate $\rho > 0$.

To construct an equilibrium, assume that all agents begin with cash $M_1 = m > 0$ and goods $q_1 = q > 0$. Suppose also that, in the market for goods, they each bid a fixed proportion

$$b(m) = am$$

of their holdings in cash, where

$$0 < b(m) \leq m + \frac{p(m)q}{1+\rho}$$

and

$$p(m) := \frac{b(m)}{q} = a \cdot \frac{m}{q}.$$

Assume that each agent puts i units of the good into production where $0 \leq i \leq b/p$, and consumes the amount $(b(m)/p) - i$. The equations (1) and (2) for the amounts of money and goods in the next period, now take the form

$$\widetilde{m} = (1+\rho)(m - b(m)) + p(m)q = (1 + \rho - \rho a)m = \tau m \tag{3}$$

and

$$\widetilde{q} = f(i) + y, \tag{4}$$

respectively, where

$$\tau := 1 + \rho - \rho a.$$

Because we are assuming that every agent has the same amount of cash and makes the same bid, equality (3) also holds for the total money supply; that is,

$$M_2 = \tau M_1.$$

Thus there need not be conservation of money in this model, and τ can be viewed as a rate of inflation (or deflation) of the money supply.

Consider now the value

$$a^* = \frac{(1+\rho)(1-\beta)}{\rho} \tag{5}$$

of a that enforces the *Fisher equation* $\tau = \beta(1+\rho)$, and let

$$\tau^* := 1 + \rho - \rho a^* = \beta(1+\rho).$$

Theorem 4 *There are two cases.*

1. *Suppose $f'(0+) < (1+\rho)/\beta$. Then there is an equilibrium in which each agent bids the proportion a^* of his cash in every period, and puts no goods into production.*
2. *Suppose $f'(0+) \geq (1+\rho)/\beta$. In this case, there exists a number $i_4 \in [0, \infty)$ such that $f'(i_4) = (1+\rho)/\beta$. There is then an equilibrium for which, in each period, every agent bids a constant proportion a^* of his cash and puts i_4 units of goods into production.*

The proof is in the Appendix.

Consider the equilibrium of case 2 in this theorem. Notice that, under this equilibrium, total goods available in each period are equal to

$$q_4 = f(i_4) + y;$$

and consumption is

$$x_4 = f(i_4) + y - i_4.$$

It is easy to see that

$$x_4 < x_1,$$

where x_1 is daily consumption for Robinson Crusoe. Also x_4 is an decreasing function of ρ, since

$$i_4 = (f')^{-1}\left(\frac{1+\rho}{\beta}\right).$$

Indeed, i_4 converges to i_1 as ρ approaches zero: again, the introduction of credit allows agents to approach Robinson Crusoe's welfare. For ρ sufficiently large, we have case 1 and there is no production.

As mentioned above, the money supply inflates or deflates geometrically in the equilibrium of Theorem 4. Indeed,

$$M_n = (\tau^*)^{n-1}\, M_1.$$

The same is true of prices, since

$$p_n \;=\; p(M_n) \;=\; \frac{a^* M_n}{q_4} \;=\; \frac{a^* \cdot (\tau^*)^{n-1} M_1}{q_4} \;=\; (\tau^*)^{n-1}\, p_1.$$

The central bank can keep prices constant (thus purging both inflation and deflation) by setting

$$1+\rho = 1/\beta$$

so that $\tau^* = 1$. However, smaller deflationary values of ρ are more efficient from the point of view of the agents' welfare.

6 Model 5: a market economy with producer firms, owner-consumer agents, saver-consumer agents, and a central bank

We continue to assume that there is a continuum of agents $\alpha \in I = [0,1]$, each of whom holds cash and bids in every period to buy goods for consumption. However, we no longer assume that these agents produce the good. Instead, there is a continuum of *firms* $\phi \in J = [0,1]$, each of which produces goods for sale in the market. The firms hold no cash, and must borrow from a central bank to purchase goods as input for production; they are owned by the above agents, who hold equal shares in all the firms and receive as income the profits earned by the firms in each period. For this reason we call these agents *owner agents*.

In addition to the owner agents, there is a continuum of *saver agents* $\gamma \in K = [0,1]$, each of whom holds cash, bids in every period to buy goods for consumption, and subsists entirely on his savings. These agents can be thought of as "retirees".

Before constructing an equilibrium we need to explain the workings of the economy in some more detail.

- Each *firm* ϕ begins every period n with goods q_n^{ϕ} that are to be sold in the market. The total amount of goods offered for sale is thus

$$Q_n = \int q_n^{\phi}\, \mathrm{d}\phi.$$

Each firm ϕ also borrows cash b_n^ϕ from a central bank, with $0 \leq b_n^\phi \leq (p_n q_n^\phi)/(1+\rho)$, where p_n is the price of the good in period n (as defined below) and $\rho > 0$ is the interest rate. The firm spends the cash b_n^ϕ to purchase the amount of goods

$$i_n^\phi = \frac{b_n^\phi}{p_n}$$

as input for production, and begins the next period with an amount of goods

$$q_{n+1}^\phi = f(i_n^\phi) + y.$$

Here $f(\cdot)$ is a *production function* which satisfies our usual assumptions, and $y \geq 0$ is the constant, deterministic endowment. During period n each firm ϕ earns the (net) profit

$$\pi_n^\phi = p_n q_n^\phi - (1+\rho) b_n^\phi,$$

since it must pay back its loan with interest. The goal of the firm is to maximize its total discounted profits

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \pi_n^\phi.$$

In a given period n the total amount of goods offered for sale by all the firms, and the total profits generated by all the firms, are

$$Q_n = \int q_n^\phi \, \mathrm{d}\phi \quad \text{and} \quad \Pi_n = \int \pi_n^\phi \, \mathrm{d}\phi,$$

respectively. The profits Π_n are distributed to the owner agents in equal shares at the end of the period.

- Consider next a typical *owner agent* α, who holds money m_n^α at the beginning of period n. The agent bids an amount of money a_n^α with $0 \leq a_n^\alpha \leq m_n^\alpha + \Pi_n/(1+\rho)$, which buys him an amount $x_n^\alpha = a_n^\alpha / p_n$ of goods. The agent begins the next period with cash

$$m_{n+1}^\alpha = (1+\rho)\left(m_n^\alpha - a_n^\alpha\right) + \Pi_n.$$

As in the other models, each agent α seeks to maximize his total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\alpha),$$

where $0 < \beta < 1$ is a given discount factor.

- Finally, consider a typical *saver agent* γ, who holds m_n^γ in cash at the start of period n. The saver bids an amount c_n^γ of cash with $0 \le c_n^\gamma \le m_n^\gamma$, which buys him a quantity $y_n^\gamma = c_n^\gamma / p_n$ of goods, and starts the next period with

$$m_{n+1}^\gamma \;=\; (1+\rho)\left(m_n^\gamma - c_n^\gamma\right)$$

in cash. If $v(\cdot)$ is his utility function, with the same properties as $u(\cdot)$, the saver agent's objective is to maximize the total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1}\, v(y_n^\gamma).$$

The total amounts of money bid in period n by the owner agents, the firms, and the saver agents, are

$$A_n = \int a_n^\alpha \,\mathrm{d}\alpha\,, \qquad B_n = \int b_n^\phi \,\mathrm{d}\phi \quad \text{and} \quad \Gamma_n = \int c_n^\gamma \,\mathrm{d}\gamma,$$

respectively. The price p_n is formed as usual, as the total bid over the total production

$$p_n = \frac{A_n + B_n + \Gamma_n}{Q_n}.$$

To construct an equilibrium, suppose that all owner agents begin with cash $M_1^A = m^A > 0$, all saver agents begin with cash $M_1^\Gamma = m^\Gamma \ge 0$, and all firms begin with goods $Q_1 = q > 0$. Thus, the total amount of cash $M_1 = M_1^A + M_1^\Gamma$ across agents, is equal to

$$m \;=\; m^A + m^\Gamma,$$

and the proportion of money held by the saver agents is

$$\nu \;=\; \frac{m^\Gamma}{m} \;=\; \frac{m^\Gamma}{m^A + m^\Gamma}, \quad \text{with} \quad 0 \le \nu < 1.$$

Suppose that the bids of the agents and firms are

$$a_1 = am, \quad b_1 = bm, \quad c_1 = cm,$$

that is, proportional to the total amount of cash, so that the price is also proportional to this amount:

$$p_1 = p(m) = \frac{(a+b+c)m}{q}.$$

Then the profit of each firm is

$$\Pi_1 = p_1 q - (1+\rho) b_1 = (a + c - \rho b)m,$$

the cash of each owner agent at the beginning of the next period is

$$M_2^A \;=\; (1+\rho)\Big(m^A - am\Big) + \Pi_1,$$

and the cash held by each saver agent is

$$M_2^{\Gamma} = (1+\rho)\left(m^{\Gamma} - cm\right).$$

Thus, the total amount of cash held by all agents at the beginning of the next period is

$$M_2 = M_2^A + M_2^{\Gamma} = \big(1+\rho-\rho(a+b+c)\big)m = \tau m,$$

where we have set

$$\tau = 1+\rho-\rho(a+b+c).$$

Define

$$r = \frac{(1+\rho)(1-\beta)}{\rho} \tag{6}$$

and notice that r is what was denoted by a^* in the context of the previous model.

Theorem 5 *There are two cases.*

1. *Suppose that $f'(0+) < (1+\rho)/\beta$. Then there is an equilibrium for which, in every period: each firm bids $b^* = 0$, inputs 0, and produces y; each owner agent bids the proportion $a^* = r - (1-\beta)\nu$ of the total money supply and consumes $[1-(\rho\nu/(1+\rho))]y$; whereas each saver agent bids the proportion $c^* = (1-\beta)\nu$ of the total money supply and consumes $[\rho\nu/(1+\rho)]y$.*
2. *Suppose that $f'(0+) \geq (1+\rho)/\beta$ so that there exists i_4 with $f'(i_4) = (1+\rho)/\beta$. Then there is an equilibrium for which, in every period: each firm inputs i_4, produces $q_4 = f(i_4) + y$, and bids the amount $b_n = b^* M_n$; each owner agent bids $a_n = a^* M_n$; and each saver agent bids $c_n = c^* M_n$. Here*

$$a^* + b^* + c^* = r, \quad b^* = \frac{r}{q_4}\cdot i_4, \quad c^* = (1-\beta)\nu \tag{7}$$

   and $M_n = M_n^A + M_n^{\Gamma}$ is the amount of cash held across agents in period n.

Furthermore, in each period n: every owner agent consumes the amount $x^* = [1-(\rho\nu/(1+\rho))]q_4 - i_4$; every saver agent consumes the amount $y^* = [\rho\nu/(1+\rho)]q_4$; whereas every firm makes $\pi^* M_n$ in profits, with $\pi^* = r - (1+\rho)b^*$.

The proof is sketched in the Appendix.

Observe that formally setting $\nu = 0$ in Theorem 5, we obtain an economy with only producer firms and owner/consumer agents.[2] For the equilibrium of Theorem 5 in such an economy, production and consumption are precisely the same as in the equilibrium of Theorem 4, where the agents both produced and consumed the good.

In the Appendix we show that the consumption and total discounted utility of the owner agents are decreasing functions of ρ in case 2 of the theorem, as they were for model 4: such agents prefer as *low* an interest rate as possible. Similarly,

[2] Of course, the proportion ν has to be strictly less than one; for otherwise there is no one to engage in productive activity, own the firms or receive their profits, and the model unravels.

the firms also prefer an interest rate as close to zero as possible, in order to maximize their profits. But the situation of the saver agents is subtler: under certain configurations of the various parameters of the model (endowment variable, discount factor, production function) they prefer as *high* an interest rate as possible; whereas under other configurations they settle on an interest rate $\rho^* \in (0, \infty)$ that uniquely maximizes their welfare.

Let

$$\tau^* = 1 + \rho - \rho(a^* + b^* + c^*).$$

Then money and prices inflate (or deflate) at rate τ^* in the equilibrium of Theorem 5 just as they did in model 4. Also, in both cases of the theorem we have $a^* + b^* = r$, so that the Fisher equation $\tau^* = \beta(1 + \rho)$ prevails again.

For a deterministic model with overlapping generations of savers, shareholders, and wage earners, it is conjectured that the savers would be eliminated and that there would be an efficient level of production in the stationary state with rate of interest equal to zero.

7 Stochastic models

Suppose that the constant y is replaced by a random variable Y in the models we have considered. It is then no longer possible to obtain closed-form solutions in general – not even for the simple Robinson Crusoe model. However, it should be possible to obtain existence results and qualitative information about equilibria. A step in this direction is the study of a stochastic version of models 1 and 4.

Appendix: some proofs

Similar techniques are used to prove all the theorems. We shall illustrate the methods by proving the simplest result, Theorem 1, and the most difficult, Theorem 4. We will also sketch the proof of Theorem 5, which is the only theorem with different types of agents.

The conventional approach would be to use Euler equations and transversality conditions, as in Stokey and Lucas (1989). This approach, as usually formulated, requires interior solutions. We will use a different technique which was introduced in Karatzas et al. (2006) and does not rely on interiority. Although quite simple, our method is unusual and we will first give a brief general description of it.

Remarks on a method of proving optimality

1. Suppose g is a real-valued function with domain D and we want to show g achieves its maximum at the point $x^* \in D$. Suppose further that h is another real-valued function with domain $\tilde{D} \supseteq D$ such that $h(x) \geq g(x)$ for all $x \in D$, $h(x^*) = g(x^*)$, and h achieves its maximum at $x^* \in \tilde{D}$. Then $\max_D g \leq \max_{\tilde{D}} h = h(x^*) = g(x^*)$. So x^* is the maximizer for g.

2. Now consider a dynamic programming problem with state space S and initial state $s \in S$. Suppose we want to show that a certain strategy σ^* is optimal at s. That is, we want to show $V(s) = R_{\sigma^*}(s)$ where $V(\cdot)$ is the optimal reward function for the dynamic programming problem and $R_{\sigma^*}(s)$ is the return from strategy σ^* at s.

Next assume that there is another dynamic programming problem with the same state space S and suppose that this new problem has optimal reward function $W(\cdot)$. Assume also that every strategy σ from the original problem is still available in the new problem and has a return $W_\sigma(s)$ at least as large as the return $R_\sigma(s)$ in the original problem. Suppose further that σ^* is optimal at s in the new problem at s and that $W_{\sigma^*}(s) = R_{\sigma^*}(s)$. Then σ^* is also optimal at s in the original problem.

This is just a special case of the previous remark with g replaced by $R_\sigma(s)$ regarded as a function of σ and h replaced by $W_\sigma(s)$.

Proof of Theorem 1

First, assume case 2 with $f'(i_1) = 1/\beta$.

We need to show that $i = i_1$ is the optimal input for Robinson Crusoe when he begins with goods $q_1 = f(i_1) + y$. This is sufficient, because Robinson Crusoe after inputting i_1 will return to the same state q_1 at the beginning of the next period.

It should be noticed that the input i_1 is always possible at q_1, because

$$q_1 \geq f(i_1) = \int_0^{i_1} f'(x)\,\mathrm{d}x \geq f'(i_1) \cdot i_1 = \frac{1}{\beta} \cdot i_1 \geq i_1.$$

We now introduce a second dynamic programming problem as in remark 2 above. First, we replace the utility function $u(\cdot)$ by the function $\tilde{u}(\cdot)$ whose graph is the tangent line to $u(\cdot)$ at $x_1 = q_1 - i_1$. More explicitly,

$$\tilde{u}(x) = \lambda x + \zeta,$$

where $\lambda = u'(x_1)$ and $\zeta = u(x_1) - x_1 u'(x_1)$. By the concavity of $u(\cdot)$, we have

$$\tilde{u}(x) \geq u(x)$$

for all x. Thus the return from any strategy will be at least as large for $\tilde{u}(\cdot)$ as for $u(\cdot)$. Notice, however, that the strategy σ^* of inputting i_1 starting from q_1 will have the same return for both utility functions because $\tilde{u}(x_1) = u(x_1)$; here x_1 is the amount consumed every day.

The second modification is to allow inputs $i \in [0, \max\{q, i_1\}]$ at every value of q. Thus there are more possible inputs and consequently more possible strategies available in the new problem. Hence, the optimal reward must be at least as large as in the original problem with its smaller utility function and fewer possible strategies. Consequently, it suffices to prove that the strategy σ^* is optimal in the modified problem at q_1, since it must then certainly be optimal for the original problem.

For proving optimality we can further simplify the problem by taking the utility function to be the identity $w(x) = x$. It is clear that optimal strategies for $w(\cdot)$ are the same as for $\tilde{u}(\cdot)$.

Let $W(q)$ be the optimal reward in our new problem at state $q \geq 0$. It satisfies the *modified Bellman equation*:

$$W(q) = \sup_{0 \leq i \leq \max\{q, i_1\}} \big[q - i + \beta\, W(f(i) + y)\big].$$

We need to show that $i = i_1$ is optimal at q_1. The following lemma states that always taking $i = i_1$, is optimal at every state q.

Lemma 1 *The strategy of inputting i_1 at every state q in every period, is optimal for the modified problem for every initial state, when $f'(0+) \geq 1/\beta$.*

Proof Let $R(q)$ be the return from the strategy. Then

$$R(q) = q - i_1 + \beta\, R(q_1) = q + k,$$

where $q_1 = f(i_1) + y$ and k is the constant determined by $k = \beta\, R(q_1) - i_1$, namely $k = (\beta q_1 - i_1)/(1 - \beta)$. It suffices to show that $R(\cdot)$ satisfies the modified Bellman equation stated above, in the form

$$R(q) = \sup_{0 \leq i \leq \max\{q, i_1\}} [\psi(i)],$$

where

$$\psi(i) = q - i + \beta R(f(i) + y) = q - i + \beta\, [f(i) + y + k].$$

Now $\psi(\cdot)$ is concave because $f(\cdot)$ is, and we have

$$\psi'(i) = -1 + \beta f'(i) = 0$$

when $i = i_1$. Thus $\psi(\cdot)$ attains its maximum at $i = i_1$. □

The proof for case 2 is now complete.

Now assume that we are in case 1 of Theorem 1, where $f'(i) \leq f'(0+) < 1/\beta$ for all $i \geq 0$. We need to show that an input of $i = 0$ is optimal at $q = y$.

Just as in case 2, there is again no harm in replacing the utility function by the identity $w(x) = x$. For this case we leave the set of possible inputs unchanged.

Lemma 2 *The strategy of inputting $i = 0$, in every period and at every state q, is optimal for the modified problem of case 1 when $f'(0+) < 1/\beta$.*

Proof Let $R(q)$ be the return from the strategy at q. Then

$$R(q) = q + \beta R(y) = q + k$$

where k is a constant. It suffices to show $R(q) = \sup_{0 \leq i \leq q} [\psi(i)]$, where

$$\psi(i) = q - i + \beta R(f(i) + y) = q - i + \beta\, [f(i) + y + k].$$

Therefore,

$$\psi'(i) = -1 + \beta f'(i) \leq 0$$

and $\psi(i)$ has its maximum at $i = 0$.

The proof of Theorem 1 is now complete. □

Proof of Theorem 4

Consider case 2 with $f'(i_4) = (1+\rho)/\beta$. Let σ^* be the strategy of bidding the amount $b = a^*m$ and putting $i = i_4$ into production in each period. This is a feasible strategy since, as in the proof of Theorem 1,

$$f(i_4) = \int_0^{i_4} f'(x)\,\mathrm{d}x \geq \frac{1+\rho}{\beta} \cdot i_4 \geq i_4.$$

Thus in equilibrium an agent who inputs i_4 in any period will produce enough goods to be able to input i_4 again in the next period. Also, the bid $b = a^*m$ satisfies

$$0 \leq a^*m < m + \frac{p(m)q}{1+\rho}.$$

This is easy to verify when we use the expression

$$p(m) = \frac{a^*m}{q}$$

and substitute for a^* from (5).

We must show that the strategy σ^* is optimal for a given agent, when all others use σ^*. To do this, suppose that the given agent has cash s and goods q; while every other agent has cash m, goods $q^* \equiv q_4 = f(i_4) + y$, and uses the strategy σ^*. The given agent faces a dynamic programming problem with optimal reward function $V(s, q, m)$ satisfying the Bellman equation

$$V(s,q,m) = \sup_{\substack{0 \leq b \leq s + \frac{p(m)q}{1+\rho} \\ 0 \leq i \leq \frac{b}{p(m)}}} \left[u\left(\frac{b}{p(m)} - i\right) + \beta\, V\big((1+\rho)(s-b) + p(m)q,\, y + f(i),\, \tau^* m\big)\right].$$

We need to show that the strategy σ^* is optimal for the given agent at states of the form (m, q^*, m), that is, when he has the same wealth and goods as all the other agents.

In period n the given agent will consume

$$\frac{b(m_n)}{p(m_n)} - i_4 = \frac{a^*\, m_n}{(a^*\, m_n)/q^*} - i_4 = q^* - i_4,$$

so that his total return is

$$\frac{1}{1-\beta} \cdot u(q^* - i_4). \tag{8}$$

- For the proof of optimality we shall modify the dynamic programming problem as we did in the proof of Theorem 1. First, we replace the utility function $u(\cdot)$ by the affine function $\tilde{u}(\cdot)$ whose graph is tangent to $u(\cdot)$ at $x^* = q^* - i_4$. Thus

$$\tilde{u}(x) = \lambda x + \zeta,$$

where $\lambda = u'(x^*)$ and $\zeta = u(x^*) - u'(x^*)x^*$. In the new problem the agent is allowed to choose at each stage any nonnegative input i such that

$$i \leq \max\{i_4, b/p(m)\}.$$

The Bellman equation for the optimal return function $W(s, q, m)$ in the new problem is

$$W(s,q,m) = \sup_{\substack{0 \leq b \leq s + \frac{p(m)q}{1+\rho} \\ 0 \leq i \leq \max\{i_4, \frac{b}{p(m)}\}}} \left[\tilde{u}\left(\frac{b}{p(m)} - q\right) + \beta\, W\Big((1+\rho)(s-b) + p(m)q,\ y + f(i),\ \tau^* m\Big)\right]. \tag{9}$$

Clearly $W(s, q, m) \geq V(s, q, m)$, because the new problem has a larger utility function and more actions. Thus, if σ^* is optimal at states (m, q^*, m) in the new problem, it must also be optimal in the original problem.

- Consider now another strategy $\tilde{\sigma}$ which, in every state (s, q, m), makes the maximum allowed bid $b = s + p(m)q/(1 + \rho)$ and always puts $i = i_4$ into production. In the first period starting from $(s_1, q_1, m_1) = (s, q, m)$, this strategy $\tilde{\sigma}$ earns utility

$$\tilde{u}\left(\frac{b}{p(m)} - i_4\right) = \tilde{u}\left(\frac{s}{p(m)} + \frac{q}{1+\rho} - i_4\right) = \lambda \cdot \left(\frac{s}{p(m)} + \frac{q}{1+\rho} - i_4\right) + \zeta.$$

At the next stage the agent is at the state

$$(s_2, q_2, m_2) = (0, q^*, \tau^* m).$$

The return from $\tilde{\sigma}$ in each period $n \geq 2$ is

$$\tilde{u}\left(\frac{q^*}{1+\rho} - i_4\right) = \lambda \cdot \left(\frac{q^*}{1+\rho} - i_4\right) + \zeta.$$

The total discounted return is

$$\begin{aligned} R(s,q,m) &= \lambda \left[\tfrac{s}{p(m)} + \tfrac{q}{1+\rho} - i_4 + \tfrac{\beta}{1-\beta}\left(\tfrac{q^*}{1+\rho} - i_4\right)\right] + \tfrac{\zeta}{1-\beta} \\ &= \lambda \left[\tfrac{s}{p(m)} + \tfrac{q}{1+\rho}\right] + k, \end{aligned} \tag{10}$$

where k is a constant.

For an initial state (m, q^*, m), trivial algebra gives, after the substitution $p(m) = a^* m/q^*$, the computation

$$\begin{aligned} R(m, q^*, m) &= \frac{1}{1-\beta} \cdot \left[\lambda(q^* - i_4) + \zeta\right] \\ &= \frac{1}{1-\beta} \cdot \tilde{u}(x^*) \\ &= \frac{1}{1-\beta} \cdot u(x^*) \\ &= \frac{1}{1-\beta} \cdot u(q^* - i_4). \end{aligned}$$

This is the same return at (m, q^*, m) as that in (8) for the strategy σ^* of the theorem. Thus σ^* is optimal at (m, q^*, m) in the original problem if $\tilde{\sigma}$ is in the modified problem.

To see that $\tilde{\sigma}$ is optimal in the new problem, we shall verify that its return function $R(\cdot, \cdot, \cdot)$ of (10) satisfies the Bellman equation (9). (This is sufficient by a result of Blackwell (1966) on positive dynamic programming. His result applies because the daily reward is bounded from below by $c = \tilde{u}(-i_4)$, and we can add $-c$ to the daily reward without affecting the optimality of any strategy.)

Let

$$\psi(b, i) = \tilde{u}\left(\frac{b}{p(m)} - i\right) + \beta\, R\Big((1+\rho)(s-b) + p(m)q,\, y + f(i),\, \tau^* m\Big).$$

We must verify that $\psi(\cdot, \cdot)$ attains its maximum over the set

$$0 \le b \le s + \frac{p(m)q}{1+\rho}, \quad 0 \le i \le \max\left\{i_4, \frac{b}{p(m)}\right\}$$

at the point

$$b = s + \frac{p(m)q}{1+\rho}, \quad i = i_4.$$

By the formula (10) for $R(\cdot, \cdot, \cdot)$ and the definition of $\tilde{u}(\cdot)$, we have

$$\begin{aligned} \psi(b, i) &= \lambda\left[\left(\frac{b}{p(m)} - i\right) + \beta\left(\frac{-(1+\rho)b}{p\,(\tau^* m)} + \frac{f(i)}{1+\rho}\right)\right] + \tilde{k} \\ &= \lambda \cdot \left[b \cdot \left\{\frac{1}{p(m)} - \frac{\beta(1+\rho)}{\tau^* p(m)}\right\} + \left\{-i + \frac{\beta}{1+\rho} \cdot f(i)\right\}\right] + \tilde{k} \\ &= \lambda \cdot \left[b \cdot 0 + \left\{-i + \frac{\beta}{1+\rho} \cdot f(i)\right\}\right] + \tilde{k}, \end{aligned}$$

where $\tilde{k}$ is a constant. Thus

$$\frac{\partial \psi}{\partial b} = 0, \quad \frac{\partial \psi}{\partial i} = -1 + \frac{\beta}{1+\rho} \cdot f'(i) = 0 \quad \text{if and only if} \quad i = i_4.$$

This completes the proof for case 2 of Theorem 4.

The proof for case 1 is similar but slightly easier, and is omitted.

Proof of Theorem 5

Assume that we are in case 2 of the theorem. Let σ be the strategy for an owner agent who bids the amount a^*m; let σ' be the strategy for a saver agent who bids the amount c^*m; and let $\tilde{\sigma}$ be the strategy for a firm that bids the amount b^*m, whenever the amount of cash held across agents is $m = m^A + m^\Gamma$. First we must check that σ, σ' and $\tilde{\sigma}$ are feasible.

When every firm follows $\tilde{\sigma}$, they each put $i^* \equiv i_4$ into production, and produce an amount $q^* \equiv q_4 = f(i_4) + y$ of goods. Thus, if every owner agent uses the same strategy σ and holds m^A in cash, and if every saver agent uses the same strategy σ' and holds m^Γ in cash, the price is

$$p(m) \;=\; (a^* + b^* + c^*)m/q^* \;=\; rm/q^*. \tag{11}$$

For σ, σ' and $\tilde{\sigma}$ to be feasible, the bids a^*m, b^*m and c^*m must satisfy the requirements

$$0 \le a^*m \le m + \frac{\pi(m)}{1+\rho}, \quad 0 \le b^*m \le \frac{p(m)q^*}{1+\rho}, \quad 0 \le c^*m \le m,$$

where each firm has profit

$$\pi(m) \;=\; p(m)q^* - (1+\rho)b^*m \;=\; \big(r - (1+\rho)b^*\big)m \;=\; \pi^* m$$

proportional to the money supply, where

$$\pi^* \;=\; r - (1+\rho)b^* \;=\; r\left(1 - \frac{(1+\rho)i^*}{f(i^*)+y}\right).$$

The third of these requirements is obviously satisfied. Clearly, both a^*m and b^*m are nonnegative. The second inequality for a^*m follows from (11), from the formula above for $\pi(m)$, and from (6), (7). The second inequality for b^*m can, by virtue of (7), be written as

$$\frac{r}{q^*} \cdot i^* \;\le\; \frac{r}{1+\rho}.$$

But

$$q^* = f(i^*) + y \ge f(i^*) \ge \int_0^{i^*} f'(u)\,\mathrm{d}u \ge f'(i^*) \cdot i^* = \frac{1+\rho}{\beta} \cdot i^* \ge (1+\rho) \cdot i^*$$

so all three strategies are feasible.

It remains to be shown that σ, σ' and $\tilde{\sigma}$ are optimal for a given owner agent, saver agent and firm, respectively.

- Consider first a given *firm* that begins with goods $q > 0$, when all other firms begin with goods q^* and play $\tilde{\sigma}$, all owner agents begin with cash $m^A > 0$ and play σ, and all saver agents begin with cash $m^\Gamma \ge 0$ and play σ'. The given firm then faces a dynamic programming problem, with optimal reward function $V(q, m)$ that satisfies the Bellman equation

$$V(q, m) = \sup_{0 \le b \le \frac{p(m)q}{1+\rho}} \left[p(m)q - b(1+\rho) + \frac{1}{1+\rho} \cdot V\left(f\left(\frac{b}{p(m)}\right) + y, \tau^* m \right) \right].$$

We must show that $\tilde{\sigma}$ is optimal for the given firm, when it begins at the same position (q^*, m) as the other firms. To show this, we extend the definition of $\tilde{\sigma}$ for all possible positions, to be the strategy that bids at position (q, m) the amount

$$\tilde{b}(q, m) = \begin{cases} b^* m, & \text{if } b^* m \le \frac{p(m)q}{1+\rho}, \\ \frac{p(m)q}{1+\rho}, & \text{if not.} \end{cases}$$

That is, $\tilde{\sigma}$ bids $b^* m$ if possible and otherwise makes the maximum possible bid. In particular, $\tilde{\sigma}$ thus defined agrees with the original definition when $q = q^*$. We shall show that $\tilde{\sigma}$ is optimal for every initial state (q, m) and, in particular, at states (q^*, m).

Let $R(q, m)$ be the total return to the single firm that starts at (q, m) and follows the plan $\tilde{\sigma}$. Then

$$R(q, m) = \begin{cases} p(m)q - b^* m(1+\rho) + \dfrac{1}{1+\rho} \cdot R(q^*, \tau^* m), & \text{if } b^* m \le \dfrac{p(m)q}{1+\rho} \\ \dfrac{1}{1+\rho} \cdot R\big(f(\dfrac{q}{1+\rho}) + y, \tau^* m\big), & \text{if not.} \end{cases} \tag{12}$$

The dynamic programming problem faced by the firm has nonnegative rewards (i.e. its profits) at each stage. Thus, by a theorem of Blackwell (1966), it suffices to show that $R(\cdot, \cdot)$ satisfies the appropriate Bellman equation, namely,

$$R(q, m) = \sup_{0 \le b \le \frac{p(m)q}{1+\rho}} \big[\psi(b) \big], \tag{13}$$

where

$$\psi(b) = p(m)q - b(1+\rho) + \frac{1}{1+\rho} \cdot R\left(f\left(\frac{b}{p(m)}\right) + y, \tau^* m \right).$$

The following lemma is helpful in the verification of (13). Its statement involves the partial derivative of the function $R(q, m)$ with respect to q. This function is piecewise smooth in q and the statement is valid wherever the derivative exists.

Lemma 6

$$\frac{\partial R}{\partial q}(q, m) = p(m) \quad \textit{if} \quad b^* m \le \frac{p(m)q}{1+\rho},$$

and

$$\frac{\partial R}{\partial q}(q, m) > p(m) \quad \textit{if} \quad b^* m > \frac{p(m)q}{1+\rho}.$$

Proof The first assertion is clear from (12). For the proof of the second, define

$$g(q) = g_1(q) = f\left(\frac{q}{1+\rho}\right) + y,$$

and set

$$g_{n+1}(q) = g(g_n(q)), \quad n \geq 1.$$

The condition $b^*m > \frac{p(m)q}{1+\rho}$ is equivalent to $q/(1+\rho) < i^*$ by (7) and (11). Now for $0 < q/(1+\rho) < i^*$, there is a unique $n \geq 1$ such that

$$g_1(q) < i^*, \ldots, g_{n-1}(q) < i^*, g_n(q) \geq i^*.$$

Also

$$\begin{aligned} R(q,m) &= \left(\frac{1}{1+\rho}\right)^n \cdot R\big(g_n(q), (\tau^*)^n m\big) \\ &= \left(\frac{1}{1+\rho}\right)^n \Big[(\tau^*)^n p(m) g_n(q) - b^*(\tau^*)^n m(1+\rho) \\ &\quad + \frac{1}{1+\rho} \cdot R\big(q^*, (\tau^*)^{n+1} m\big)\Big] \\ &= (\beta)^n p(m) g_n(q) + k, \end{aligned}$$

where $k = k(m)$ does not depend on q and we have used the equality $\tau^* = \beta(1+\rho)$. Now

$$g'(q) = \frac{1}{1+\rho} \cdot f'\left(\frac{q}{1+\rho}\right) > \frac{1}{1+\rho}\frac{1+\rho}{\beta} = \frac{1}{\beta}$$

and an easy induction on n shows that

$$g_n'(q) > \frac{1}{\beta^n}.$$

The second assertion follows. □

In order to verify (13), we need to show that the function $\psi(b)$ attains its maximum at $b = \tilde{b}(q,m)$. If $b > b^*m$, then $b/p(m) > b^*m/p(m) = i^*$ and $f(b/p(m)) + y \geq f(i^*) + y \geq (1+\rho)i^*$. By the lemma,

$$\begin{aligned} \psi'(b) &= -(1+\rho) + \frac{1}{(1+\rho)p(m)} f'\left(\frac{b}{p(m)}\right) p(\tau^* m) \\ &= -(1+\rho) + \frac{1}{1+\rho} f'\left(\frac{b}{p(m)}\right) \tau^* \\ &< -(1+\rho) + \frac{1}{1+\rho} \cdot \beta(1+\rho)\frac{1+\rho}{\beta} = 0 \end{aligned}$$

so the maximizing bid never exceeds b^*m.

Now suppose that $b < b^*m$ so that $b/p(m) < b^*m/p(m) = i^*$ and $f'(b/p(m)) > f'(i^*) = (1+\rho)/\beta$. The inequality in the calculation above now reverses, to give $\psi'(b) > 0$. Thus the maximum occurs at either b^*m or the maximum allowed value of b, whichever is smaller. The proof that $\tilde{\sigma}$ is optimal for the given firm is now complete.

- Consider next the situation of a given *owner agent* with cash s, when all other owner agents have cash m^A each and play σ; all firms have goods q^* each and play $\tilde{\sigma}$; and all saver agents have cash m^Γ each and play σ'. We need to show that σ is optimal for the given agent when $s = m^A$. The proof is similar to that of Theorem 4.

  The given agent with cash s faces a dynamic programming problem with optimal reward function $W(s, m)$ satisfying the Bellman equation

$$W(s,m) = \sup_{0 \le a \le \frac{s+\pi(m)}{(1+\rho)}} \left[u\left(\frac{a}{p(m)}\right) + \beta\, W\big((1+\rho)(s-a) + \pi(m),\ \tau^* m\big)\right].$$

  As in the proof of Theorem 4, we introduce another dynamic programming problem with a larger utility function $\tilde{u}(\cdot)$ and a larger action set. Let

$$x^* = \frac{a^*m}{p(m)} = \frac{a^*}{a^*+b^*+c^*}\cdot q^* = \frac{a^*q^*}{r} = \left(1 - \frac{\rho\nu}{1+\rho}\right) q^* - i^*$$

  and define the utility function $\tilde{u}(\cdot)$ as

$$\tilde{u}(x) = \lambda x + \zeta,$$

  where $\lambda = u'(x^*)$ and $\zeta = u(x^*) - x^*u'(x^*)$. In the new problem the agent at position (s, m) is allowed to select any action a such that

$$s - \beta(1-\nu)\, m + \frac{\pi(m)}{1+\rho} \le a \le s + \frac{\pi(m)}{1+\rho}.$$

  The plan that bids

$$a = a(s,m) = s - \beta(1-\nu)\, m + \frac{\pi(m)}{1+\rho}$$

  in each position (s, m) can be shown to be optimal in the new problem by the same methods used to prove Theorem 4. Also this plan coincides with σ if $s = m^A$; indeed, it is readily checked from (7), (6) that $a(m^A, m) = a^*m$. It follows that the strategy σ is optimal in the original problem for the given owner agent, when he begins in the same position as the other owner agents.
- Finally, let us consider the situation of a given *saver agent* with cash s, when all other saver agents have cash m^Γ each and play σ'; all firms have goods q^* each and play $\tilde{\sigma}$; and all owner agents have cash m^A each and play σ. We need to show that σ' is optimal for the given agent when $s = m^\Gamma$.

  The given agent with cash s faces a dynamic programming problem with optimal reward function $U(s, m)$ satisfying the Bellman equation

$$U(s,m) = \sup_{0 \le c \le s} \left[v\left(\frac{c}{p(m)}\right) + \beta\, U\big((1+\rho)(s-c)\big)\right].$$

As in the proof of Theorem 4, we introduce another dynamic programming problem with a larger utility function $\hat{v}(\cdot)$ and a larger action set. Let

$$y^* = \frac{c^* m}{p(m)} = \frac{a^* q^*}{a^* + b^* + c^*} = \frac{\rho\nu}{1+\rho} q^*$$

and define the utility function

$$\hat{v}(x) = \mu x + \eta,$$

where $\mu = v'(y^*)$ and $\eta = v(y^*) - y^* v'(y^*)$. In the new problem the agent at position (s, m) is allowed to select any action c with

$$s - \beta\nu m \le c \le s.$$

The plan that bids

$$c = c(s, m) = s - \beta\nu m$$

in each position (s, m) can be shown to be optimal in the new problem by the same methods used to prove Theorem 4. This plan coincides with σ if $s = m^{\Gamma}$; and it is easily checked that $c(m^{\Gamma}, m) = c^* m$.

It follows that the strategy σ' is optimal in the original problem for the given saver agent, when he begins in the same position as the other saver agents. The proof of case 2 is now complete. The proof of case 1 is similar and is omitted.

Sensitivity analysis

Let us make the blanket assumptions $y > 0$ and $f'(0+) = \infty$, which ensconce us firmly within case 2 of Theorem 5, and try to analyze the welfare of the agents as a function of the prevailing interest rate ρ.

- The optimal total discounted reward of the *owner agents* is

$$W^*(\rho) \equiv W\left(m^A, m\right) = \sum_{n=1}^{\infty} \beta^{n-1} u\left(x^*(\rho)\right) = \frac{u\left(x^*(\rho)\right)}{1-\beta};$$

here

$$x^*(\rho) = \left(1 - \frac{\rho\nu}{1+\rho}\right)\left[f\left(I\left(\frac{1+\rho}{\beta}\right)\right) + y\right] - I\left(\frac{1+\rho}{\beta}\right)$$

is the owner agents' daily optimal consumption, explicitly displayed as a function of the interest rate $\rho > 0$, and $I = \left(f'\right)^{-1}$ is the inverse of the marginal production function. We claim that $\rho \mapsto x^*(\rho)$, thus also $\rho \mapsto W^*(\rho)$, are decreasing: *the owner agents prefer as low an interest rate as possible.* Indeed, a straightforward computation yields

$$\frac{\partial}{\partial\rho} x^*(\rho) = \frac{1-\beta+\rho(1-\nu)}{\beta^2} I'\left(\frac{1+\rho}{\beta}\right) - \frac{\nu}{(1+\rho)^2}\left[f\left(I\left(\frac{1+\rho}{\beta}\right)\right) + y\right] < 0,$$

since $I'(\cdot) < 0$.

- The optimal total discounted profits for a typical *firm* are

$$V^*(\rho) \equiv V\left(q^*, m\right) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \cdot \pi^* M_n,$$

where $M_n = m_1 \tau^{n-1} = m_1\big(\beta(1+\rho)\big)^{n-1}$ is the total money supply on day n. Therefore,

$$\frac{V^*(\rho)}{m_1} = \frac{\pi^*}{1-\beta} = \frac{r}{1-\beta}\left[1 - \frac{(1+\rho)\, i^*}{f(i^*)+y}\right] \\ = \frac{1+\rho}{\rho}\left[1 - \beta\,\Psi\left(\frac{1+\rho}{\beta}\right)\right]$$

where

$$\Psi(\xi) = \frac{\xi I(\xi)}{f(I(\xi))+y} = \left.\frac{i f'(i)}{f(i)+y}\right|_{i=I(\xi)}.$$

Let us observe that $\lim_{\xi\to\infty} \Psi(\xi) = 0$. This is because we have $f(i) > i\, f'(i)$ from concavity, thus

$$0 < \frac{i f'(i)}{f(i)+y} < \frac{f(i)}{f(i)+y} \longrightarrow 0 \quad \text{as} \quad i \downarrow 0.$$

It is then fairly clear that

$$\lim_{\rho\to\infty} V^*(\rho) = m_1 \cdot \lim_{\rho\to\infty} \left(1+\frac{1}{\rho}\right) \cdot \big[\, 1 - \beta\,\Psi(\infty)\, \big] = m_1.$$

On the other hand, notice that we have $1-\beta < 1-\beta\,\Psi(\xi)$, so $\underline{\lim}_{\xi\downarrow(1/\beta)} \big(1 - \beta\,\Psi(\xi)\big) \geq 1-\beta > 0$. This implies

$$\lim_{\rho\downarrow 0} V^*(\rho) = m_1 \cdot \lim_{\rho\downarrow 0} \left\{\left(1+\frac{1}{\rho}\right)\left[1-\beta\,\Psi\left(\frac{1+\rho}{\beta}\right)\right]\right\} = \infty,$$

and makes clear that *the firms should prefer an interest rate as close to* 0 *as they can get*.
- The situation is subtler for the *saver agents*. Their optimal total discounted reward is

$$U^*(\rho) \equiv U\left(m^\Gamma, m\right) = \sum_{n=1}^{\infty} \beta^{n-1}\, v\left(y^*(\rho)\right) = \frac{v\left(y^*(\rho)\right)}{1-\beta},$$

where

$$y^*(\rho) = \frac{\rho\nu}{1+\rho}\, q^* = \frac{\rho\nu}{1+\rho}\left[f\left(I\left(\frac{1+\rho}{\beta}\right)\right)+y\right]$$

is their daily optimal consumption.

We shall study an example that illustrates the behavior of this function. With $f(i) = 2\sqrt{i}$ we have $f'(i) = 1/\sqrt{i}$, $I(\xi) = \xi^{-2}$ and $f\big(I(\xi)\big) = 2/\xi$, thus

$$\frac{y^*(\rho)}{\nu} = \frac{\rho}{1+\rho}\left(y + \frac{2\beta}{1+\rho}\right), \quad \frac{1}{\nu} \cdot \frac{\partial}{\partial\rho} y^*(\rho) = \frac{(y+2\beta) + \rho(y-2\beta)}{(1+\rho)^3}.$$

If $y \geq 2\beta$, the function $y^*(\cdot)$ is increasing and the saver agents prefer as high an interest rate as possible.

If $0 < y < 2\beta$ on the other hand, the function $y^*(\cdot)$ attains its maximum over $(0, \infty)$ at the point

$$\rho^* = \frac{2\beta + y}{2\beta - y},$$

which is then best from the saver agents' point of view.

References

Blackwell, D.: Positive dynamic programming. Proc 5th Berkeley Symp Math Stat Probab **1**, 415–418 (1966)

Bobenrieth, E., Bobenrieth, J., Wright, B.: A commodity price process with a unique continuous distribution having infinite mean. Econometrica **70**, 1213–1219 (2002)

Geanakoplos, J., Karatzas, I., Shubik, M., Sudderth, W.D.: A stochastic economy with production and a central bank. (in preparation, 2006)

Hakansson, N.H.: Optimal investment and consumption strategies under risk for a class of utility functions. Econometrica **38**, 587–607 (1969)

Karatzas, I., Shubik, M., Sudderth, W., Geanakoplos, J.: The inflationary bias of real uncertainty and the harmonic Fisher equation. Econ Theory **28**, 481–512 (2006)

Levhari, O., Srinivasan, T.N.: Optimal savings under uncertainty. Rev Econ Stud **26**, 155–163 (1969)

Ljungquist, L., Sargent, T.J.: Recursive macroeconomic theory. Cambridge: MIT (2000)

Phelps, E.S.: The accumulation of risky capital: a sequential utility analysis. Econometrica **30**, 729–743 (1962)

Ramsey, F.P.: A mathematical theory of saving. Econ J **38**, 543–559 (1928)

Samuelson, P.A.: Lifetime portfolio selection by dynamic stochastic programming. Rev Econ Stat **51**, 239–246 (1969)

Samuelson, P.A.: Stochastic speculative price. Proc Natl Acad Sci **68**, 335–337 (1971)

Scheinkman, J.A., Schechtman, J.: A simple competitive model with production and storage. Rev Econ Stud **L**, 427–441 (1983)

Stokey, N.L., Lucas, R.E.: Recursive methods in economic dynamics. Cambridge: Harvard University Press (1989)