Designing R and S charts

Data are $N(\mu, \sigma^2)$. Rational group size $n$. $R$ and $S$ are the range and the standard deviation of rational group.

$E(R) = d_2 \sigma; \ E(S) = c_4 \sigma; \ sd(R) = d_3 \sigma; \ sd(S) = \sqrt{1 - c_4^2} \sigma$

These give centerlines and upper and lower control limits.

Probability distribution of $R$ known but not familiar. Can be used to get probability limits

For $S$ chart, $(n-1)S^2 / \sigma^2$ is $\chi^2$ with $n-1$ degrees of freedom. This means $\alpha$ probability limits given by

$LCL = \sigma \sqrt{\frac{\chi^2_\alpha}{n-1}}; \ UCL = \sigma \sqrt{\frac{\chi^2_{1-\alpha}}{n-1}}$

Note. Unlike familiar testing setup, we have $\alpha$ in each tail, not combined.
Ex. – aspirin, use $\sigma = 2$, $n = 8$, $\alpha = 0.001$

$d_2 = 2.8472$, $d_3 = 0.8198$, $c_4 = 0.9650$

$d_2 \pm 3d_3$ is 0.3878 and 5.3066.

**Three sigma**

R  CL, LCL, UCL = 5.69, 0.78, 10.62  
C  CL, LCL, UCL = 1.93, 0.36, 3.50

**Probability**

From tables or software, lower and upper 0.001 points of R and S are 0.8348, 5.8227, 0.2924, 1.8640 giving

R  CL, LCL, UCL = 5.69, 1.67, 11.64  
S  CL, LCL, UCL = 1.93, 0.58, 3.72

**Check**

From $\chi^2$ tables with 7 df, 0.001 and 0.9999 points are 0.5985, 24.322, giving same results.
What is IC behavior of 3-sigma limits? For $n=8$, S chart 3sigma limits are 0.18, 1.75. These correspond to $\chi^2$ values of $7 \times 0.18^2$ and $7 \times 1.75^2 = 0.23, 21.44$. From printed chi-squared table, area on left is tiny. Area on right between 0.005 and 0.001.

Software gives 5e-6 and 0.0044. Areas both very different from 0.00135.
Out of control behavior

\[ R/\sigma \] and \[ S/\sigma \] are standardized – like Z scores. Distribution depends only on \( n \). Upper control limits on R and S are of the form \[ R/\sigma \] or \[ S/\sigma > L \] only depends on \( n \).

If IC value of \( \sigma \) is \( \sigma_0 \) then UCL for S is \[ S > L \sigma_0 \] or, standardizing S,
\[ S/\sigma > L \sigma_0/\sigma. \]
So the \( \text{Pr}[\text{signal}] \) is the IC \( \text{Pr}[\text{Signal}] \) for a chart with UCL recalled to \( L \sigma_0/\sigma \).

Same reasoning hold for LCL, and for R charts. OOC behavior of scale charts can be figures from IC behavior of charts with different control limits.