Some Cusum Theory

In control, $X$ has distribution $f_0(x)$. At some unknown instant, it may go out of control, shifting to $f_1(x)$. The optimal diagnostic for this is the cusum with recursion

$$C_n = \max(0, C_{n-1} + Z_n), \text{ where } Z_n = \log \left( \frac{f_1(X_n)}{f_0(X_n)} \right) = \log[f_1(X_n)] - \log[f_0(X_n)]$$

Control of normal means

Example – in-control is $N(\mu_0, \sigma^2)$, out of control is $N(\mu_1, \sigma^2)$. $f_i(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - \mu_i)^2}{2\sigma^2} \right)$.

$$\log[f_1(X_n)] = -0.5 \log(2\pi) - \log \sigma - \frac{(X_n - \mu_1)^2}{2\sigma^2} \text{ and after a bit of simplification}$$

$$Z_n = \frac{\mu_1 - \mu_0}{\sigma^2} \left[ X_n - \frac{\mu_1 + \mu_0}{2} \right]$$

The $\frac{\mu_1 - \mu_0}{\sigma^2}$ is a constant that applies to the whole cusum, so what we actually cusum is

$$Z_n = \left[ X_n - \frac{\mu_1 + \mu_0}{2} \right] = \left[ X_n - \mu_0 - \frac{\mu_1 - \mu_0}{2} \right] \text{ the same as we have been doing.}$$

But note that if $\mu_1 < \mu_0$, the multiplier reverses the cusum sign, so technically the cusum is

$$C_n = \max\left(0, C_{n-1} + X_n - \frac{\mu_0 + \mu_1}{2}\right) \quad \mu_1 > \mu_0$$

$$= \max\left(0, C_{n-1} - X_n + \frac{\mu_0 + \mu_1}{2}\right) \quad \mu_1 < \mu_0 \quad \text{as in the book}$$

But many users like the chart for a downward shift to show downward (negative) numbers and use

$$C_n = \max\left(0, C_{n-1} + X_n - \frac{\mu_0 + \mu_1}{2}\right) \quad \mu_1 > \mu_0$$

$$= \min\left(0, C_{n-1} + X_n - \frac{\mu_0 + \mu_1}{2}\right) \quad \mu_1 < \mu_0$$
Control of normal variances

Here, in-control is \( N(\mu, \sigma_0^2) \), out of control is \( N(\mu, \sigma_1^2) \). \( f_i(x) = \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left( -\frac{(x - \mu)^2}{2\sigma_i^2} \right) \).

\[
\log[f_i(X_n)] = -0.5 \log(2\pi) - \log \sigma_i - \frac{(X_n - \mu)^2}{2\sigma_i^2}
\]

and after some simplification

\[
Z_n = \log\left(\frac{\sigma_0}{\sigma_1}\right) + (X_n - \mu)^2 \cdot \frac{1}{2} \left[ \frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} \right]
\]

After some rearranging, this is proportional to

\[
Z_n = (X_n - \mu)^2 - K
\]

where the reference value \( K \) is

\[
K = \frac{2\sigma_0^2 \sigma_1^2 \log(\sigma_0 / \sigma_1)}{\sigma_0^2 - \sigma_1^2}
\]

As with the normal mean, we’ll show downward shifts as negative numbers, defining

\[
V_n = \max\left( 0, V_{n-1} + (X_n - \mu)^2 - K \right) \quad \sigma_1 > \sigma_0
\]

\[
= \min\left( 0, V_{n-1} + (X_n - \mu)^2 - K \right) \quad \sigma_1 < \sigma_0
\]
Combined cusums for location and scale

Putting these together, IC, $X \sim N(\mu_0, \sigma_0^2)$. The process could go out of control by (a) mean goes to $\mu_1$, and/or (b) sd goes to $\sigma_2$.

To cover all bases, we set up cusums for upward and downward shift in mean; and upward and downward shifts in standard deviation: four in all.

Set each to an IC ARL four times what you want for the overall scheme. For example, to get combo IC ARL of 250, set each cusum to a IC ARL of 1000.

Be aware that some people don’t like this idea. They say that if the mean shifts, it will affect the scale cusum and increase its false alarm rate. This is true, but perhaps not important as long as you are using the location cusum to check location. As the location cusum is the optima; diagnostic for a location shift, you should see it there long before you see its shadow in the scale cusum.

In any event, if the variance goes up, that increases the false alarm rate of any location chart. A doubling of the standard deviation has the same effect as halving the control limits of the location chart and so increasing its false alarm rate, so a shift of one type affecting a chart designed for shift of another type is quite common.

To protect yourself from misinterpretation, after a combination control scheme signals, it is a good idea to calculate the mean and standard deviation of the readings following the estimates last in-control and see how they match up with the in-control values.