Stat 5101 (Geyer) Spring 2019
Homework Assignment 6
Due Wednesday, March 13, 2019

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

6-1. The function

$$f(x) = 2x, \qquad 0 < x < 1$$

is a PDF. Suppose X is a random variable having this PDF.

(a) Calculate $E(X)$.

(b) Calculate $E(X^2)$.

(c) Calculate $\operatorname{var}(X)$.

6-2. The function

$$f(x, y) = x + y, \qquad 0 < x < 1, \ 0 < y < 1$$

is a PDF. Suppose (X, Y) is a random vector having this PDF.

(a) Calculate $E(X)$.

(b) Calculate $\operatorname{var}(X)$.

(c) Calculate $\operatorname{cov}(X, Y)$.

(By symmetry, $E(Y) = E(X)$ and $\operatorname{var}(Y) = \operatorname{var}(X)$ so we do not need to calculate them.)

6-3. Suppose (X, Y) is a continuous random vector having PDF f. Say for each of the following definitions of f whether X and Y are independent or not.

(a) $f(x, y) = 4xy$, $0 < x < 1$, $0 < y < 1$.

(b) $f(x, y) = 8xy$, $0 < x < y < 1$.

(c) $f(x, y) = 144(x - 1/2)^2(y - 1/2)^2$, $0 < x < 1$, $0 < y < 1$.

(d) $f(x, y) = 288(x - 1/2)^2(y - 1/2)^2$, $0 < x < y < 1$.

6-4. Suppose X is a continuous random variable having PDF

$$f(x) = \begin{cases} 1 + x, & -1 \le x < 0 \\ 1 - x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

(b) Find $E(X^2)$.

(c) Find $\operatorname{var}(X^2)$.

Hint: Since the PDF has a case-splitting formula, you must split integrals into pieces

$$E\{g(X)\} = \int_{-1}^{0} g(x) f(x)\, dx + \int_{0}^{1} g(x) f(x)\, dx$$

such that the PDF is defined by one formula for each piece.

6-5. Suppose X is a continuous random variable having the $\operatorname{Exp}(\lambda)$ distribution. Write $\mu = E(X)$.

(a) Find $E\{(X - \mu)^3\}$.

(b) Find $E\{(X - \mu)^4\}$.

6-6. Suppose (X, Y) is a continuous random vector having PDF

$$f(x, y) = 2, \qquad 0 < x < y < 1$$

(a) Find $E(X)$

(b) Find $E(Y)$

(c) Find $E(X^2)$

(d) Find $E(Y^2)$

(e) Find $E(XY)$

(f) Find $\operatorname{var}(X)$

(g) Find $\operatorname{var}(Y)$

(h) Find $\operatorname{cov}(X, Y)$

Hint: the limits of integration are a bit tricky.

$$E\{g(X,Y)\} = 2 \int_0^1 \int_0^y g(x,y)\,dx\,dy = 2 \int_0^1 \int_x^1 g(x,y)\,dy\,dx$$

6-7. Suppose X is a continuous random variable having the same PDF as in problem 6-4. Find its distribution function. Be sure to define the DF on the whole real line.

6-8. Suppose X is a continuous random variable having DF

$$F(x) = \begin{cases} 0, & x \le 1 \\ 1 - 1/x, & x > 1 \end{cases}$$

Find its PDF. Define the PDF on the whole real line.

6-9. Suppose U has the $\text{Unif}(0,1)$ distribution. What is the PDF of

$$Y = -\frac{1}{\lambda} \log(U)$$

6-10. Suppose X has the $\text{Unif}(-1,1)$ distribution. What is the PDF of

$$Y = X^2$$

6-11. Suppose (X,Y) has the uniform distribution on the disk

$$\{\, (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \,\}$$

What is the PDF of the random vector (R,T) which is (X,Y) expressed in polar coordinates?

Hint: The map from (r,t) to (x,y) is given by

$$\begin{aligned} x &= r\cos(t) \\ y &= r\sin(t) \end{aligned}$$