Stat 5102 (Geyer) Spring 2015
Homework Assignment 9
Due Wednesday, April 15, 2015

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

9-1. Suppose X is $\text{Bin}(n, p)$ and the prior distribution for p is flat (a proper prior, since p is bounded).

(a) Find the posterior distribution for p.

(b) Find the mean of the posterior distribution for p.

(c) Find the standard deviation of the posterior distribution for p.

(d) Find the mode of the posterior distribution for p.

(e) In the case $x = 0$, $n = 10$, find the posterior median for p.

9-2. Suppose $X_1, \ldots, X_n$ are IID $\text{Exp}(\lambda)$ and the prior distribution for λ is flat (an improper prior). The posterior distribution for λ was found in problem 8-10 (a).

(a) Find the mean of the posterior distribution for λ.

(b) Find the standard deviation of the posterior distribution for λ.

(c) Find the mode of the posterior distribution for λ.

(d) In the case $\bar{x}_n = 23.7$, $n = 10$, find the posterior median for λ.

9-3. Find the Jeffreys prior for the $\text{NegBin}(r, p)$ distribution, considering r fixed and known and p the unknown parameter. It is proper or improper?

9-4. Find the Jeffreys prior for the $\text{Gam}(\alpha, \lambda)$ distribution, where α is known and λ unknown, so we want a prior distribution for λ. It is proper or improper?

9-5. Find the posterior mean and variance of μ when the data are IID normal and the prior is a general normal-gamma prior. Say for which values of the hyperparameters of the prior the posterior mean and variance of μ exist.

9-6. Suppose $X_1, \ldots, X_n$ are IID $\mathcal{N}(\mu, 4)$, the prior distribution for μ is $\mathcal{N}(10, 9)$, and the sample mean of a sample of size 10 is $\overline{X}_n = 12$. Calculate a 90% HPD region for μ (note not 95%).

9-7. Suppose $X_1, \ldots, X_n$ are IID $\mathcal{N}(\mu, \lambda^{-1})$, the prior distribution for (μ, λ) is the conjugate normal-gamma prior with

$$\begin{aligned} \lambda &\sim \text{Gam}(3, 3) \\ \mu \mid \lambda &\sim \mathcal{N}(10, 16\lambda^{-1}) \end{aligned}$$

the sample mean of a sample of size 15 is $\overline{X}_n = 12$ and the sample variance is $S_n^2 = 50$ (note not V_n).

(a) Calculate a 95% HPD region for μ.

(b) Calculate the exact frequentist 95% confidence interval for μ.

9-8. Suppose $X_1, \ldots, X_n$ are IID $\text{Exp}(\lambda)$ and the prior distribution for λ is $\text{Gam}(3, 3)$.

(a) Calculate the posterior probabilities of the events

$$\begin{aligned} H_0 &\colon \lambda \geq 1 \\ H_1 &\colon \lambda < 1 \end{aligned}$$

when $n = 4$ and $\bar{x}_n = 1.9$.

(b) Calculate the prior probabilities of the same events.

(c) Calculate the Bayes factor

$$\frac{\Pr(H_0 \mid \mathbf{x})}{\Pr(H_1 \mid \mathbf{x})} \cdot \frac{\Pr(H_1)}{\Pr(H_0)}$$

(d) Calculate an exact frequentist P-value for these hypotheses based on the exact sampling distribution of $X_1 + \cdots + X_n$.

9-9. Suppose $X_1, \ldots, X_n$ are IID $\text{Exp}(\lambda)$. In this problem we are interested in the hypotheses (models)

$$\begin{aligned} m_1 = H_0 &\colon \lambda = 1 \\ m_2 = H_1 &\colon \lambda \neq 1 \end{aligned}$$

Suppose the prior distribution for λ given model m_2 is $\text{Gam}(3, 3)$. The prior distribution for λ given model m_1 is concentrated at the point $\lambda = 1$. Suppose $n = 4$ and $\bar{x}_n = 1.9$. Calculate the Bayes factor (model 1 over model 2). Hint: For both models, proceed as if the data were $X_1 + \cdots + X_n$.