

Stat 5102 (Geyer) Spring 2012  
Homework Assignment 7  
Due Wednesday, March 21, 2012

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

**7-1.** Suppose  $X_1, \dots, X_n$  are IID Cauchy( $\mu, \sigma$ ) and  $\sigma = 1$  is known. We wish to do maximum likelihood estimation, which cannot be done in closed form, so you must use R. One needs a good estimate of the location parameter to use as a starting point for the optimization. The location parameter is the center of symmetry and also the median. Thus the sample median is a good starting point. Data for the problem are at the URL

<http://www.stat.umn.edu/geyer/5102/data/prob7-1.txt>

- (a) Find the MLE for these data.
- (b) Find the observed Fisher information evaluated at the MLE.
- (c) Find an asymptotic 95% confidence interval for the parameter  $\mu$ .

**7-2.** Suppose  $x_1, \dots, x_n$  are known numbers (not random), and we observe random variables  $Y_1, \dots, Y_n$  that are independent but *not* identically distributed random variables having distributions

$$Y_i \sim \mathcal{N}(\alpha + \beta x_i, \sigma^2),$$

where  $\alpha, \beta$ , and  $\sigma^2$  are unknown parameters.

- (a) Write down the log likelihood for the parameters  $\alpha, \beta$ , and  $\varphi = \sigma^2$ .
- (b) Find the maximum likelihood estimates of these parameters.
- (c) Find the expected Fisher information matrix for these parameters.

**Caution:** In taking expectations remember only the  $Y_i$  are random. The  $x_i$  are known constants.

**7-3.** Suppose  $X_1, \dots, X_n$  are IID Laplace( $\mu, \sigma$ ) and both parameters are considered unknown. We found the log likelihood in problem 6-4. Find the MLE of  $\mu$  and  $\sigma$ .

**Hint:** For each fixed value of  $\sigma$  show that the sample median is the MLE of  $\mu$ , the argument being as in problem 6-13. Call that MLE  $\hat{\mu}_n$ , now plug this estimate into the log likelihood obtaining a function

$$\sigma \mapsto l_n(\hat{\mu}_n, \sigma) \tag{*}$$

which is a twice differentiable function of  $\sigma$ . Find a point where the first derivative of (\*) is zero, and show that it is a local maximizer. That point is the MLE for  $\sigma$ .

**7-4.** Suppose  $X_1, \dots, X_n$  are IID Cauchy( $\mu, \sigma$ ) and both parameters are unknown. We wish to do maximum likelihood estimation, which cannot be done in closed form, so you must use R. One needs a good estimate of both parameters to use as a starting point for the optimization. As in problem 7-1 the sample median is a good starting point for  $\mu$ . In 5101 problem 8-10(c) we found that the interquartile range (IQR) of the standard Cauchy distribution is 2. Thus the IQR of the Cauchy( $\mu, \sigma$ ) distribution is  $2\sigma$ , and half the IQR is a good starting point for  $\sigma$ . The R function `IQR` estimates the IQR. Data for the problem are at the URL

<http://www.stat.umn.edu/geyer/5102/data/prob7-1.txt>

- (a) Find the MLE vector for these data.
- (b) Find the observed Fisher information matrix evaluated at the MLE.
- (c) Find asymptotic 95% confidence intervals for the parameters  $\mu$  and  $\sigma$ . Do not adjust to obtain 95% simultaneous coverage.

**7-5.** Suppose  $X_1, \dots, X_n$  are IID Gam( $\alpha, \lambda$ ) and  $\lambda = 1$  is known. Suppose  $n = 50$ , the MLE is  $\hat{\alpha}_n = 2.73$  and we wish to do test of the hypotheses

$$H_0: \alpha = 2.5$$

$$H_1: \alpha \geq 2.5$$

Find the asymptotic  $P$ -value for this test using the standardized MLE as the test statistic. You will have to use R to calculate Fisher information. Interpret the  $P$ -value.

## Review Problems from Last Year's Tests

**7-6.** Suppose  $X_1, \dots, X_n$  are IID Gam( $\alpha, \lambda$ ), where  $\alpha$  is known and  $\lambda$  is unknown.

- (a) Find the log likelihood for  $\lambda$ .
- (b) Find the maximum likelihood estimate (MLE) for  $\lambda$ .
- (c) Show that your MLE is the unique global maximizer of the log likelihood.
- (d) Find the expected Fisher information for  $\lambda$ .

**7-7.** Suppose  $X_1, \dots, X_n$  are IID having PDF

$$f_{\theta}(x) = \theta x^{\theta-1}, \quad 0 < x < 1$$

where  $\theta > 0$  is an unknown parameter.

- (a) Find the log likelihood for  $\theta$ .
- (b) Find the maximum likelihood estimate (MLE) for  $\theta$ .
- (c) Find the asymptotic distribution of your MLE.
- (d) Find an asymptotic 95% confidence interval for  $\theta$ . (Hint: The 0.95 quantile of the standard normal distribution is 1.645, and the 0.975 quantile of the standard normal distribution is 1.96.)

**7-8.** Suppose  $X$  is  $\text{Exp}(\lambda)$ , where  $\lambda$  is an unknown parameter. We have only one observation.

- (a) Describe how to do an exact (not approximate) test of the hypotheses

$$H_0: \lambda = \lambda_0$$

$$H_1: \lambda < \lambda_0$$

where  $\lambda_0$  is a specified number (the value of  $\lambda$  hypothesized under  $H_0$ ). Give a formula for the  $P$ -value of the test, an expression in terms of functions you can find on a calculator. (Hint: Consider the relationship between  $E(X)$  and  $\lambda$ .)

- (b) Calculate the  $P$ -value when  $\lambda_0 = 1$  and  $x = 3.7$ .
- (c) Interpret the  $P$ -value. What does it say about whether  $\lambda$  is larger or smaller than  $\lambda_0$ ? What does it say about the statistical significance of this conclusion?