Stat 5102 (Geyer) Spring 2012
Homework Assignment 2
Due Wednesday, February 1, 2012

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

2-1. Suppose X_1, X_2, $\ldots$ are IID Unif$(0, \theta)$. As usual $X_{(n)}$ denotes the n-th order statistic, which is the maximum of the X_i.

(a) Show that

$$X_{(n)} \xrightarrow{P} \theta, \qquad \text{as } n \to \infty.$$

(b) Show that

$$n\bigl(\theta - X_{(n)}\bigr) \xrightarrow{\mathcal{D}} \text{Exp}(1/\theta), \qquad \text{as } n \to \infty.$$

Hints This is a rare problem (the only one of its kind we will meet in this course) when we can't use the LLN or the CLT to get convergence in probability and convergence in distribution results (obvious because the problem is not about $\overline{X}_n$ and the asymptotic distribution we seek isn't normal). Thus we need to derive convergence in distribution directly from the characterization as convergence of distribution functions (5101 deck 6, slide 4), that is,

$$X_n \xrightarrow{\mathcal{D}} X$$

if and only if F_n is the DF of X_n and F is the DF of X and

$$F_n(x) \to F(x), \qquad \text{whenever } F \text{ is continuous at } x.$$

Hint for Part (a): Show that the DF of $X_{(n)}$ converges to the DF of the constant random variable θ. (Why does this do the job?)
Hint for Part (b): Define

$$Y_n = n\bigl(\theta - X_{(n)}\bigr)$$

(the random variable we're trying to get an asymptotic distribution for). Derive its DF $F_{Y_n}(y)$. What you need to show is that

$$F_{Y_n}(y) \to F(y), \qquad \text{for all } y$$

where F is the DF of the Exp$(1/\theta)$ distribution. The fact from calculus (which you may use without proving it)

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

is useful in this.

You can derive the DF of Y_n from the DF of $X_{(n)}$, which is given on slide 106, 5102 deck 1.

2-2. Suppose X_1, ..., X_n are IID $\mathcal{N}(\mu, \sigma^2)$. What is the probability that $|\overline{X}_n - \mu| > 2S_n/\sqrt{n}$ if $n = 10$?

2-3. Suppose X_1, ..., X_n are IID $\mathcal{N}(\mu, \sigma^2)$. What is the probability that $S_n^2 > 2\sigma^2$ if $n = 10$?

2-4. Show if X has k-th central moment

$$\mu_k = E\{(X - \mu)^k\}$$

where, as usual, $\mu = E(X)$, then $Y = a + bX$ has k-th central moment $b^k \mu_k$.

2-5. What is the asymptotic distribution of the variance V_n of the empirical distribution for an IID Exp(λ) sample?

2-6. Suppose X is standard normal (so $\mu_X = 0$ and $\sigma_X = 1$).

(a) What is $P(|X| > 2\sigma_X)$?

In contrast, suppose X has a $t(3)$ distribution (so $\mu_X = 0$ and the variance σ_X^2 is given by Problem 1-7 on assignment 1.

(b) Now what is $P(|X| > 2\sigma_X)$?

2-7. Suppose Y is a random scalar having mean μ and variance σ^2 and $\mathbf{Z}$ is a random vector with IID components Z_i having mean zero and variance τ^2, and suppose also that Y is independent of $\mathbf{Z}$. Define $\mathbf{X} = Y + \mathbf{Z}$ (that is, $\mathbf{X}$ has components $X_i = Y + Z_i$).

(a) What are the mean vector and variance matrix of $\mathbf{X}$?

(b) Suppose $\mathbf{X}_1$, $\mathbf{X}_2$, ... is an IID sequence of random vectors having the same distribution as $\mathbf{X}$. What is the asymptotic distribution of $\overline{\mathbf{X}}_n$?

2-8. Show that S_n^2 and V_n have the same asymptotic distribution. **Hint:** Slutsky's theorem.

2-9. Find the asymptotic distribution of S_n. Do not assume normality of the population distribution. **Hint:** delta method.

2-10. Find the asymptotic distribution of the sample median of an IID sample from

(a) a Cauchy(μ, σ) distribution, and

(b) a Laplace(μ, σ) distribution.

PDF for both are on the brand name distributions handout.