Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

9-1. Suppose $X$ is Bin($n, p$) and the prior distribution for $p$ is flat (a proper prior, since $p$ is bounded).

(a) Find the posterior distribution for $p$.
(b) Find the mean of the posterior distribution for $p$.
(c) Find the standard deviation of the posterior distribution for $p$.
(d) Find the mode of the posterior distribution for $p$.
(e) In the case $x = 0, n = 10$, find the posterior median for $p$.

9-2. Suppose $X_1, \ldots, X_n$ are IID Exp($\lambda$) and the prior distribution for $\lambda$ is flat (an improper prior). The posterior distribution for $\lambda$ was found in problem 8-10 (a).

(a) Find the mean of the posterior distribution for $\lambda$.
(b) Find the standard deviation of the posterior distribution for $\lambda$.
(c) Find the mode of the posterior distribution for $\lambda$.
(d) In the case $\bar{x}_n = 23.7, n = 10$, find the posterior median for $\lambda$.

9-3. Find the Jeffreys prior for the NegBin($r, p$) distribution, considering $r$ fixed and known and $p$ the unknown parameter. It is proper or improper?

9-4. Find the Jeffreys prior for the Gam($\alpha, \lambda$) distribution, where $\alpha$ is known and $\lambda$ unknown, so we want a prior distribution for $\lambda$. It is proper or improper?

9-5. Find the posterior mean and variance of $\mu$ when the data are IID normal and the prior is a general normal-gamma prior. Say for which values of the hyperparameters of the prior the posterior mean and variance of $\mu$ exist.

9-6. Suppose $X_1, \ldots, X_n$ are IID $\mathcal{N}(\mu, 4)$, the prior distribution for $\mu$ is $\mathcal{N}(10, 9)$, and the sample mean of a sample of size 10 is $\bar{X}_n = 12$. Calculate a 90% HPD region for $\mu$ (note not 95%).
9-7. Suppose $X_1, \ldots, X_n$ are IID $\mathcal{N}(\mu, \lambda^{-1})$, the prior distribution for $(\mu, \lambda)$ is the conjugate normal-gamma prior with 
\[
\lambda \sim \text{Gam}(3, 3) \\
\mu \mid \lambda \sim \mathcal{N}(10, 16\lambda^{-1})
\]
the sample mean of a sample of size 15 is $\bar{X}_n = 12$ and the sample variance is $S^2_n = 50$ (note not $V_n$).
(a) Calculate a 95% HPD region for $\mu$.
(b) Calculate the exact frequentist 95% confidence interval for $\mu$.

9-8. Suppose $X_1, \ldots, X_n$ are IID $\text{Exp}(\lambda)$ and the prior distribution for $\lambda$ is $\text{Gam}(3, 3)$.
(a) Calculate the posterior probabilities of the events 
\[
H_0 : \lambda \geq 1 \\
H_1 : \lambda < 1
\]
when $n = 4$ and $\bar{x}_n = 1.9$.
(b) Calculate the prior probabilities of the same events.
(c) Calculate the Bayes factor 
\[
\frac{\Pr(H_0 \mid x)}{\Pr(H_1 \mid x)} \cdot \frac{\Pr(H_1)}{\Pr(H_0)}
\]
(d) Calculate an exact frequentist $P$-value for these hypotheses based on the exact sampling distribution of $X_1 + \cdots + X_n$.

9-9. Suppose $X_1, \ldots, X_n$ are IID $\text{Exp}(\lambda)$. In this problem we are interested in the hypotheses (models)
\[
m_1 = H_0 : \lambda = 1 \\
m_2 = H_1 : \lambda \neq 1
\]
Suppose the prior distribution for $\lambda$ given model $m_2$ is $\text{Gam}(3, 3)$. The prior distribution for $\lambda$ given model $m_1$ is concentrated at the point $\lambda = 1$. Suppose $n = 4$ and $\bar{x}_n = 1.9$. Calculate the Bayes factor (model 1 over model 2). Hint: For both models, proceed as if the data were $X_1 + \cdots + X_n$. 
