Additional Problems

1. Suppose the probability density function f of a random variable X is defined by

$$f(x) = \begin{cases} \frac{1}{x^2}, & x > 1 \\ 0, & x < 1 \end{cases}$$

(the value of f at $x = 1$ does not matter).

Find the cumulative distribution function of X.

Be sure to define your answer on the whole real line.

2. Suppose the cumulative distribution function F of a random variable X is defined by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^4, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Find the probability density function of X.

Be sure to define your answer on the whole real line.

3. Suppose the joint probability density function f of random variables X and Y is defined by

$$f(x, y) = \begin{cases} 2y \exp(-xy - 2y), & x > 0 \text{ and } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal probability density function of X.

(b) Find the marginal probability density function of Y.

(c) Are X and Y independent random variables?

4. Suppose X has probability density

$$f(x) = \frac{1}{2\sqrt{x}}, \qquad 0 < x < 1$$

(note the domain).

(a) For what positive integers k does X^k have expectation?

(b) Calculate $E(X^k)$ for the positive integers k such that the expectation exists.

5. Suppose X has probability density

$$f(x) = \frac{c}{(\nu + x^2)^{(\nu+1)/2}}, \qquad -\infty < x < +\infty$$

where c is a positive constant and ν is another positive constant.

(a) Show that f is integrable (so some positive c exists that makes f a probability density).

(b) For what positive integers k does $E(X^k)$ exist? (This depends on the value of ν.)

6. Suppose X, Y, and Z are random variables such that

$$E(X \mid Y, Z) = Y$$
$$\operatorname{var}(X \mid Y, Z) = Z$$

Find the (unconditional) mean and variance of X in terms of the means, variances, and covariance of Y and Z.

(**Hint:** You must use the iterated conditional mean and variance formulas: Theorem 4.7.1 and Problem 4.7.11 in DeGroot and Schervish.)

7. Do Problem 4.8.15 in DeGroot and Schervish, but instead answer the following questions

(a) If a student's score on the mathematics test is x, what predicted value of his or her score on the music test has the smallest M. S. E.? [Same as part (a) of the question in DeGroot and Schervish, except the number 0.8 is replaced by the variable x. Of course, plugging in $x = 0.8$ gives the answer in the back of the book.]

(b) If a student's score on the music test is y, what predicted value of his or her score on the music test has the smallest M. A. E.? [Same as part (b) of the question in DeGroot and Schervish, except the number 1/3 is replaced by the variable y. Of course, plugging in $y = 1/3$ gives the answer in the back of the book.]

8. A brand of raisin bran averages 84.2 raisins per box. The boxes are filled from large bins of well mixed raisin bran. What is the standard deviation of the number of raisins per box?

9. Let X be the number of winners of a lottery. If we assume that players pick their lottery numbers at random, then their choices are i. i. d. random variables and X is binomially distributed. Since the mean number of winners is small, the Poisson approximation is very good. Hence we may assume that X has a Poisson distribution with mean np, where n is the total number of tickets sold and p is the probability of any particular ticket winning.

Because of our independence assumption, what other players do is independent of what you do. Hence the conditional distribution of the number of other winners given that you win is also Poisson with mean np. If you are lucky enough to win, you must split the prize with X other winners. You win $A/(X+1)$ where A is the total prize money. Thus

$$E\left\{\frac{A}{X+1}\right\}$$

is your expected winnings given that you win. Calculate this expectation (as a function of the constants A, n, and p).

10. Suppose three random variables have joint p. d. f. given by

$$f(x,y,z) = 6, \qquad 0 < x < y < z < 1$$

(a) Find the three two-dimensional marginal p. d. f.'s, specifying the support for each. Note: this problem is not symmetric in the three variables so there are three different functional forms.

(b) Find the three one-dimensional marginal p. d. f.'s, specifying the support for each. Note: again, there are three different functional forms.

(c) Find the three conditional p. d. f.'s that have one variable in front of the bar and the other two variables behind the bar. Indicate the support of each, meaning the set of values of the variable in front of the bar that have nonzero density (which depends on the values of the variables behind the bar).

(d) Find the three conditional p. d. f.'s that have two variables in front of the bar and the other variable behind the bar. Indicate the support of each, meaning the set of values of the variables in front of the bar that have nonzero density (which depends on the value of the variable behind the bar).

11. For each collection of marginal and conditional p. d. f.'s below, indicate whether when multiplied together they give the joint distribution of the three variables. Explain your answers.

(a) $f(x|y,z)$ and $f(y|z)$ and $f(z)$

(b) $f(x|y,z)$ and $f(y,z)$

(c) $f(x|y,z)$ and $f(y|x,z)$ and $f(z|x,y)$

(d) $f(x|y,z)$ and $f(z|y)$ and $f(z)$

Review Problems

1. The function

$$f(x) = \frac{60}{11}(x - 2x^2 + 3x^3 - 2x^4), \qquad 0 < x < 1$$

is the probability density function of a random variable X.

(a) Calculate $E(X)$.

(b) Calculate $\text{var}(X)$.

Answers. (a) 6 / 11. (b) 45 / 847. Not asked but a useful auxiliary result: $E(X^2) = 27/77$.

2. Suppose $\text{var}(X) = 3$.

(a) Calculate $\text{var}(4X + 5)$.

(b) Calculate $\text{sd}(4X + 5)$.

Answers. (a) 48. (b) $4\sqrt{3}$.