Stat 5101 Final Exam **December 17, 2002**

Name _____ Student ID _____

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. and also the handout on brand name distributions. You may use a calculator, but shouldn't need to. Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from.

1. [25 pts.] The function

$$f(x, y) = \frac{2}{2 - \log 2} \cdot \frac{x + y}{1 + y} \qquad 0 < x < 1,\ 0 < y < 1$$

is the joint probability density function of random variables X and Y. Find $E(X \mid Y)$ and $\text{var}(X \mid Y)$.

If it helps, the same joint density was used for the first question on the second midterm and the conditional density was found to be

$$g_1(x \mid y) = \frac{x + y}{\frac{1}{2} + y}, \qquad 0 < x < 1.$$

2. [25 pts.] Suppose that X and Y are random variables, the marginal distribution of X is Geometric(p) distribution and the conditional distribution of Y given X is Binomial(X, p). Find $E(Y)$ and $\text{var}(Y)$. Hint: Use the iterated expectation and variance formulas.

3. [20 pts.] Suppose buses passing on Washington avenue during evening rush hour can be considered a homogenous Poisson process with intensity of 1.5 buses per minute.

   (a) If X is the number of buses that go by during a specified 10-minute interval during evening rush hour, what is the distribution of the random variable X? What are $E(X)$ and $\text{sd}(X)$?

   (b) If Y is the time elapsed between my arrival on Washington avenue during evening rush hour and the first bus to go by after my arrival, what is the distribution of the random variable Y? What are $E(Y)$ and $\text{sd}(Y)$?

(c) If Z is the time elapsed between my arrival on Washington avenue during evening rush hour and the tenth bus to go by after my arrival, what is the distribution of the random variable Z? What are $E(Z)$ and $\mathrm{sd}(Z)$?

4. [20 pts.] Suppose X and Y are independent random variables having distributions Binomial(n, p) and Gamma(α, β), respectively

   (d) Under what conditions on the parameters n, p, α, and β does $X+Y$ have an approximately normal distribution?

   (e) What is this normal approximation (that is, what are its mean and variance in terms of n, p, α, and β)?

   (f) Suppose $n = 100$, $p = 0.1$, $\alpha = 50$, and $\beta = 5$. Approximate the probability $\Pr(X + Y > 25)$ using this normal approximation. A table of the normal distribution is the last page of the test form.

5. [25 pts.] Suppose X and Y are dependent random variables, the marginal distribution of X is Beta(α, β) and the conditional distribution of Y given X is Binomial(n, X).

   (g) What is the conditional distribution of X given Y? You may either specify it by density or by "brand name" and parameters.

   (h) What is $E(X \mid Y)$?

   (i) What is $\mathrm{var}(X \mid Y)$?

6. [20 pts.] Suppose X and Y are independent and identically distributed Gamma(α, β) random variables and suppose $\alpha > 1$. Calculate $E(X/Y)$. You do not have to prove that the expectation exists (I'm telling you it does). Simplify your answer so that it does not contain gamma functions. **Note:** $E(X/Y)$ is *not* $E(X \mid Y)$. What we want here is the unconditional expectation of the fraction X/Y, *not* the conditional expectation of X given Y.

7. [25 pts.] A random variable has probability density function f defined by

$$f(x) = \begin{cases} (1 + 2x + 3x^2)/3, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

   Find the distribution function, defining your answer on the whole real line.

8. [20 pts.] There exists a constant c such that the function

$$f(x) = c \cdot \frac{(\sin x)^2 + x^2}{(1 + x^2 + x^4 + x^{16})}, \qquad -\infty < x < \infty$$

is a probability density function. For what positive integers k does $E(X^k)$ exist?

9. [20 pts.] Suppose X is a Beta(α, β) random variable and $Y = X/(1 - X)$. Find the probability density function of Y, giving both a formula for the density and the domain of definition.