Stat 5101 (Geyer) Fall 2020
Homework Assignment 10
Due Wednesday, December 2, 2020

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

10-1. Suppose (X_1, X_2) is a bivariate normal random vector, and assume it is nondegenerate. Write its PDF in terms of the mean vector and variance matrix. Then rewrite its PDF in terms of new parameters, which are

$$
\begin{aligned}
E(X_1) &= \mu_1 \\
E(X_2) &= \mu_2 \\
\operatorname{sd}(X_1) &= \sigma_1 \\
\operatorname{sd}(X_2) &= \sigma_2 \\
\operatorname{cov}(X_1, X_2) &= \rho \sigma_1 \sigma_2
\end{aligned}
$$

Then simplify your expression for the PDF so it contains no matrices, no matrix inverses, determinants, or matrix multiplication.

Hint: the inverse of a matrix

$$
\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
$$

can be done by Cramer's rule obtaining

$$
\mathbf{A}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}
$$

assuming $\mathbf{A}$ is invertible, which it is if $\det(\mathbf{A})$ is not zero.

10-2. Suppose (X_1, X_2) is a nondegenerate bivariate normal random vector. Calculate the conditional PDF of X_1 given X_2 not using the theory developed in class. Just use conditional = joint/marginal, where the joint is the PDF you found in problem 10-1.

10-3. Suppose (X_1, X_2) is a nondegenerate bivariate normal random vector.

(a) What is the best prediction of X_1 that is a function of X_2 when "best" means minimizing expected squared prediction error?

(b) What is the best prediction of X_1 that is a function of X_2 when "best" means minimizing expected absolute prediction error?

10-4. If $\mathbf{X}$ is a standard normal random vector and $\mathbf{O}$ is an orthogonal matrix of the same dimension, show that $\mathbf{O}^T \mathbf{X}$ is a standard normal random vector.

10-5. Suppose $\mathbf{Y}$ is a nondegenerate multivariate normal random vector having mean vector $\boldsymbol{\mu}$ and variance matrix $\mathbf{M}$. Show that the random scalar

$$(\mathbf{Y} - \boldsymbol{\mu})^T \mathbf{M}^{-1} (\mathbf{Y} - \boldsymbol{\mu}) \tag{$*$}$$

has the chi-square distribution with degrees of freedom that is the dimension of $\mathbf{Y}$.

Hint: write $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{M}^{1/2}\mathbf{X}$, where the square root matrix $\mathbf{M}^{1/2}$ is as defined on slide 110, deck 5 of the course slides and $\mathbf{X}$ is a standard multivariate normal random vector. Slide 111 of that deck shows that any normal random vector $\mathbf{Y}$ can be written this way. Then plug that into equation $(*)$ above obtaining $\mathbf{X}^T\mathbf{X}$, and consider the distribution of that.

10-6. If X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution, show that $E\{X^{\beta_1}(1 - X)^{\beta_2}\}$ exists if and only if $\beta_1 > -\alpha_1$ and $\beta_2 > -\alpha_2$.

10-7. Let

$$f(x) = ce^{-|x|}, \qquad -\infty < x < \infty.$$

(a) Show that there exists a constant c such that f is a probability distribution and determine the c that makes it so.

(b) For what positive β does $E(|X|^\beta)$ exist?

(c) Find $E(X)$.

(d) Find $\text{var}(X)$.

(e) Write the PDFs of location-scale family containing f using the mean for the location parameter and the standard deviation for the scale parameter.

10-8. Let

$$f(x) = c \cdot \frac{1}{2 + \cos(x) + \cos(2x) + x^2 + x^6}, \qquad -\infty < x < \infty$$

(a) Show that there exists a constant c such that f is a probability distribution.

(b) For what positive real β does $E(|X|^\beta)$ exist?

Hint: $-1 \le \cos(x) \le 1$ for all x and $\cos(0) = 1$. You can't integrate f. You have to use the theory about existence.

10-9. Suppose $\alpha_1 > 0$ and $\alpha_2 > 0$. Let

$$f(x) = c \cdot \sin(x) x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1}, \qquad 0 < x < 1.$$

(a) Show that there exists a constant c such that f is a probability distribution.

(b) For what real β_1 and β_2 does $E\{X^{\beta_1}(1-X)^{\beta_2}\}$ exist?

Hint:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

and $0 \leq \sin(x) \leq 1$ for $0 \leq x \leq \pi$. You can't integrate f. You have to use the theory about existence.

Review Problems from Previous Tests

10-10. (a) For what real numbers α does there exist a constant $c(\alpha)$ such that the function

$$f_\alpha(x) = \frac{c(\alpha)}{x^\alpha}, \qquad x > 1$$

is a PDF?

(b) Suppose α is such that f_α is a PDF. If X is a random variable having this PDF, for what positive real numbers β does $E(X^\beta)$ exist?

10-11. Suppose telephone calls arriving at a call center between 10:00 am and 11:00 am on a weekday can be modeled as a Poisson process with rate 10 calls per minute.

(a) What is the mean and standard deviation of the number of the number of calls arriving in a 20 minute interval?

(b) What is the mean and standard deviation of the time interval between now and the time of arrival of the next call to arrive?

(c) What is the mean and standard deviation of the time interval between now and the time of arrival of the fifth call to arrive (after now)?