Stat 5101 (Geyer) Fall 2020
Homework Assignment 9
Due Wednesday, November 25, 2020

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

9-1. Suppose

$$
\begin{aligned}
E(Y \mid X) &= X \\
\operatorname{var}(Y \mid X) &= 3X^2
\end{aligned}
$$

and suppose the marginal distribution of X is $\mathcal{N}(\mu, \sigma^2)$.

(a) Find $E(Y)$.

(b) Find $\operatorname{var}(Y)$.

9-2. Suppose X_1, ..., X_N are IID having mean μ and variance σ^2 where N is a $\operatorname{Poi}(\lambda)$ random variable independent of all of the X_i. Let

$$
Y = \sum_{i=1}^{N} X_i,
$$

with the convention that $N = 0$ implies $Y = 0$.

(a) Find $E(Y)$.

(b) Find $\operatorname{var}(Y)$.

9-3. Suppose that the conditional distribution of Y given X is $\operatorname{Poi}(X)$, and suppose that the marginal distribution of X is $\operatorname{Gam}(\alpha, \lambda)$. Show that the marginal distribution of Y is a negative binomial distribution in the extended sense discussed in the brand name distributions handout in which the shape parameter need not be an integer. Identify the parameters of this negative binomial distribution (which are functions of α and λ).

9-4. Suppose that X and Y are independent Poisson random variables having means μ_1 and μ_2, respectively. Show that the conditional distribution of X given $X + Y$ is binomial. Identify the parameters of this binomial distribution (which are functions of μ_1 and μ_2 and $X + Y$).

9-5. Suppose that X has the $\operatorname{Geo}(p)$ distribution, show that the conditional distribution of $X - k$ given $X \geq k$ also has the $\operatorname{Geo}(p)$ distribution.

9-6. Suppose service times of customers in line at a bank teller are IID $\text{Exp}(\lambda)$ random variables. Suppose when you arrive there are nine customers in line in front of you (ten customers including you). What is the mean and standard deviation of the time until you are served (including your service time)?

9-7. Suppose cars crossing the midpoint of the new 35W bridge about 8:00 am on a weekday can be modeled as a Poisson process with rate 20 cars per minute. What is the mean and standard deviation of the number of cars crossing in 5 minutes?

9-8. Suppose the random vector $(X_1, \ldots, X_k)$ has the $\text{Multi}(n, \mathbf{p})$ distribution and $i \neq j$.

(a) Find $E(X_i - X_j)$.

(b) Find $\text{var}(X_i - X_j)$.

9-9. Suppose the random vector $(X_1, \ldots, X_k)$ has a multinomial distribution, and suppose we "factor" the joint PMF as the product of univariate marginals and conditionals

$$f(x_1, \ldots, x_k) = f(x_1 \mid x_2 \ldots, x_k) f(x_2 \mid x_3 \ldots, x_k) \cdots f(x_{k-1} \mid x_k) f(x_k)$$

Show that each of the univariate conditionals except for $f(x_1 \mid x_2 \ldots x_k)$ is binomial and the univariate marginal is binomial. Give the parameters of each in terms of the parameters of the joint distribution. What is the distribution of X_1 given X_2, ..., X_k?

Hint: no calculation necessary. All of the answer is determined by theory on the slides.

9-10. Suppose X_1, ..., X_k are independent Poisson random variables. Show that the conditional distribution of the random vector $(X_1, \ldots, X_k)$ given $X_1 + \ldots + X_k$ is multinomial. Give the parameters of this multinomial in terms of the parameters of the Poisson distributions.

9-11. Suppose the marginal distribution of the random variable N is Poisson having mean $\mu > 0$, and suppose the conditional distribution of the random vector $(X_1, \ldots, X_k)$ given N is $\text{Multi}(N, \mathbf{p})$. Show that the marginal distribution of $(X_1, \ldots, X_k)$ has independent components and that the distribution of each component is Poisson. Calculate $E(X_i)$, which is the parameter of its marginal Poisson distribution.

Hint: the degeneracy of the conditional distribution of $(X_1, \ldots, X_k)$ given N makes this tricky. Eliminate X_1 by writing it as a function of X_2, ..., X_k and N. Then write the joint distribution of X_2, ..., X_k and N as the product of marginal and conditional. Now eliminate N by writing it as a function of X_1, ..., X_k to obtain the joint PMF of X_1, ..., X_k.