Stat 5101 (Geyer) Fall 2020
Homework Assignment 7
Due Wednesday, November 4, 2020

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

7-1. If X has the $\text{Gam}(\alpha, \lambda)$ distribution, we calculated in class that

$$E(X^\beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\lambda^\beta}.$$

(a) Find $E(X^2)$

(b) Find $\text{var}(X)$.

None of your answers should contain gamma functions (use the gamma function recursion formula to simplify).

7-2. If X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution, show that

$$E\{X^{\beta_1}(1 - X)^{\beta_2}\} = \frac{\Gamma(\alpha_1 + \alpha_2)\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_1 + \alpha_2 + \beta_1 + \beta_2)}$$

Hint: use the fact that the PDF of the beta distribution integrates to one, just like we did for the gamma distribution. You may ignore the issue of when the integral exists (it exists when $\beta_1 > -\alpha_1$ and $\beta_2 > -\alpha_2$, but we don't know how to prove that yet).

7-3. Suppose X has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution.

(a) Find $E(X)$.

(b) Find $E(X^2)$.

(c) Find $\text{var}(X)$.

None of your answers should contain gamma functions (use the gamma function recursion formula to simplify). Hint: use the result of problem 7-2.

7-4. If X and Y are independent gamma random variables with shape parameters α_1 and α_2, respectively, and the same rate parameter, then we proved in class that $V = X/(X+Y)$ has the $\text{Beta}(\alpha_1, \alpha_2)$ distribution. Suppose instead we were interested in the random variable $W = X/Y$.

(a) Express W as a function of V.

(b) Show that this function is invertible (that is, V is also a function of W).

(c) Find the PDF of W. Be sure to give the domain as well as the formula.

7-5. Suppose the random vector (X, Y) has the uniform distribution on the square $(0,1)^2$. Find the PDF of the random vector (U, V) where

$$U = X + Y$$
$$V = \frac{X}{X+Y}$$

Be sure to give the domain as well as the formula. Express the domain clearly enough so that you can draw a correct picture of it. Hint: the support of (U, V) is

$$\{ (u, v) \in \mathbb{R}^2 : 0 < uv < 1 \text{ and } 0 < u(1-v) < 1 \}.$$

The problem is to express clearly how this constrains u and v.

7-6. Suppose X has a distribution that is symmetric about a point a. Show that, if the mean of X exists, then $E(X) = a$.

7-7. Suppose X has a distribution that is symmetric about a point a. Show that every odd central moment that exists is zero.

7-8. Show that the moment generating function of the standard normal distribution is $t \mapsto e^{t^2/2}$. Hint: use the fact that a general normal PDF integrates to one.

7-9. Suppose X has the standard normal distribution.

(a) Find $E(X^2)$.

(b) Find $E(X^4)$.

Hint: use the result of problem 7-8.

7-10. Suppose X has a distribution that is symmetric about a point a, and let $Y = \mu + \sigma X$ with $\sigma > 0$. Let $\mu_{X,k}$ and $\mu_{Y,k}$ denote the k-th central moments of X and Y, respectively. Show that

$$\mu_{Y,k} = \sigma^k \mu_{X,k}$$

7-11. Suppose X has the $\mathcal{N}(\mu, \sigma^2)$ distribution.

(a) Find $E(X)$.

(b) Find $\operatorname{var}(X)$.

(c) Find $E\{(X - \mu)^3\}$.

(d) Find $E\{(X - \mu)^4\}$.

Hint: use the results of problems 7-6, 7-7, 7-9, and 7-10.

7-12. Suppose (X, Y) has the PDF

$$f(x, y) = \frac{1 + x}{\pi}, \qquad x^2 + y^2 < 1.$$

Find the PDF of the random vector (R, T) which is (X, Y) expressed in polar coordinates, that is,

$$\begin{aligned} X &= R \cos(T) \\ Y &= R \sin(T) \end{aligned}$$

7-13. In class, we found out that the $\operatorname{Unif}(a, b)$ family of distributions is a location-scale family, but a and b are not the mean and variance. Suppose we wanted to parametrize the family so that the location parameter is the mean μ and the scale parameter is the standard deviation σ. Then what is the form of the PDFs of the family?

7-14. Suppose X_1, ..., X_n are IID standard normal random variables, then

$$R^2 = \sum_{i=1}^{n} X_i^2$$

has the chi-square distribution on n degrees of freedom.

(a) Find the PDF of R.

(b) Find $E(R)$.

(c) Find $E(R^2)$.

7-15. Show that, if a random variable X is symmetric about a point a, then a is a median of X. Comment: we already know that if the mean exists, then a is a mean of X. Hence the mean, median, and center of symmetry are all the same for a symmetric distribution, if the mean exists.

7-16. If Φ and Φ^{-1} denote the distribution and quantile functions, respectively, for a standard normal random variable, what are the distribution and quantile functions for a $\mathcal{N}(\mu, \sigma^2)$ random variable?

7-17. Suppose U is a Unif$(0, 1)$ random variable, and suppose F is a distribution function and G the corresponding quantile function. Show that the random variable $G(U)$ has F as its DF. Hint: for $0 < u < 1$ show that $G(u) \leq x$ if and only if $F(x) \geq u$. This requires use of the fact that DF are right continuous.