Stat 5101 (Geyer) Fall 2020
Homework Assignment 5
Due Wednesday, October 21, 2020

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

5-1. Suppose X_1, X_2, $\ldots$ are IID random variables having mean μ and variance τ^2. For each $i \geq 1$ define

$$Y_i = \sum_{j=1}^{5} X_{i+j}$$

Then Y_1, Y_2, $\ldots$ is called a *moving average of order 5* time series, MA(5) for short. It is a weakly stationary time series. (So far this repeats the setup for Problem 4-6.) If

$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i$$

show that

$$\overline{Y}_n - 5\mu = O_p(n^{-1/2}).$$

5-2. Suppose that X_1, X_2, $\ldots$ are IID random variables. For some subset A of the real numbers, define

$$Y_i = I_A(X_i), \qquad i = 1, 2, \ldots$$

and

$$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

Show that $\overline{Y}_n$ converges in probability to $\Pr(X_i \in A)$, and in fact

$$\overline{Y}_n - \Pr(X_i \in A) = O_p(n^{-1/2}).$$

Also show that $\overline{Y}_n$ is the fraction of X_1, $\ldots$, X_n that lie in A.

5-3. Define

$$\begin{aligned} S &= \{1, 2, \ldots, 10\} \\ A &= \{\, s \in S : s < 5 \,\} \\ B &= \{\, s \in S : 2 < s < 8 \,\} \\ C &= \{10\} \end{aligned}$$

(a) Calculate A^c where complements are taken relative to S.

(b) Calculate $A \cup B$.

(c) Calculate $A \cap B$.

(d) Calculate $A \cap C$.

5-4. Suppose A and B are independent events. Show that if A and B are also mutually exclusive events, then either $\Pr(A) = 0$ or $\Pr(B) = 0$.

5-5. Suppose A and B are independent events, and $\Pr(A) = \Pr(B) = p$. Calculate $\Pr(A \cup B)$.

5-6. Suppose X is a random variable and Y is a constant random variable. Show that X and Y are independent.

5-7. Some people define a geometric random variable to be the time of the first success in a Bernoulli process. If X is a $\text{Geo}(p)$ random variable by our definition, then $Y = X + 1$ is a $\text{Geo}(p)$ random variable by their definition.

(a) Give the PMF of Y. Be careful to indicate the possible values of Y.

(b) Calculate $E(Y)$.

(c) Calculate $\text{var}(Y)$.

Now that this problem is done, we will always use our definition of $\text{Geo}(p)$, never their definition.

5-8. Let X be a $\text{Bin}(n, p)$ random variable.

(a) Calculate $E(e^{tX})$.

(b) Explain why $t \mapsto E(e^{tX})$ is a moment generating function.

(c) Calculate $E(X)$ using the method of moment generating functions.

(d) Calculate $\text{var}(X)$ using the method of moment generating functions.

5-9. Suppose X and Y are independent binomial random variables with the same success probability but possibly different sample sizes. Show that the distribution of $X + Y$ is binomial using the convolution formula, that is, evaluate

$$\sum_x f(x) g(z - x)$$

and show that it is a binomial PMF. Hint: use the fact that probabilities sum to one. (This is problem 4-8 done the hard way.)

5-10. Suppose the number of raisins in a box of raisin bran can be considered a Poisson random variable, and suppose the mean number of raisins in a box is 100. What is the standard deviation of the number of raisins in a box?

5-11. Assuming that people pick lottery numbers at random and independently (which is not a bad strategy because you don't want to pick a popular number), the exact distribution of the number of winners is binomial, but since the number of players is very large and the probability of winning is very small, the Poisson distribution is a good approximation. If you win the lottery, you have to split the jackpot with the other winners. By the independence assumption, the distribution of the number of other winners given that you win is still Poisson with the same mean. Suppose the size of the jackpot is a known constant a and the distribution of the number of winners is $\text{Poi}(\mu)$. Then your expected winnings are

$$E\left(\frac{a}{1+X}\right).$$

Evaluate this expectation.