Stat 5101 (Geyer) Fall 2020
Homework Assignment 4
Due Wednesday, October 7, 2020

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

4-1. If U, V, X, and Y are any random variables, show that

$$\operatorname{cov}(U + V, X + Y) = \operatorname{cov}(U, X) + \operatorname{cov}(V, X) + \operatorname{cov}(U, Y) + \operatorname{cov}(V, Y)$$

4-2. Suppose X_1, X_2, X_3 are IID with mean μ and variance σ^2. Calculate the mean vector and variance matrix of the random vector

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_1 - X_2 \\ X_2 - X_3 \\ X_3 - X_1 \end{pmatrix}$$

4-3. Suppose X and Y are independent random variables, with means μ_X and μ_Y, respectively, and variances σ_X^2 and σ_Y^2, respectively. Calculate

$$E(X^2 Y^2)$$

in terms of μ_X, μ_Y, σ_X^2, and σ_Y^2.

4-4. Suppose 6 balls that are indistinguishable except for color are placed in an urn and suppose 3 balls are red and 3 are white. Suppose 2 balls are drawn. What is the probability the one is red and the other white under each of the following conditions?

(a) The 2 balls constitute a random sample with replacement from the urn.

(b) The 2 balls constitute a random sample without replacement from the urn.

What is the probability the both balls are red under each of the following conditions?

(c) The 2 balls constitute a random sample with replacement from the urn.

(d) The 2 balls constitute a random sample without replacement from the urn.

4-5. If X_1, …, X_n are exchangeable random variables, show that

$$\operatorname{cov}(X_1, X_2) \geq -\frac{\operatorname{var}(X_1)}{n - 1}$$

Hint: consider the variance of $X_1 + \cdots + X_n$.

4-6. Suppose X_1, X_2, ... are IID random variables having mean μ and variance τ^2. For each $i \geq 1$ define

$$Y_i = \sum_{j=1}^{5} X_{i+j}$$

Then Y_1, Y_2, ... is called a *moving average of order 5* time series, MA(5) for short. It is a weakly stationary time series.

(a) Calculate $E(Y_i)$.

(b) Calculate $\operatorname{var}(Y_i)$.

(c) Calculate $\operatorname{cov}(Y_i, Y_{i+k})$, for $k = 1, 2, \ldots$.

4-7. Suppose X_1 and X_2 are IID random variables that are uniformly distributed on the set $\{1, 2, 3, 4, 5\}$.

(a) Find the PMF of the random vector $\mathbf{Y} = (X_1, X_1 + X_2)$.

(b) Are the components of $\mathbf{Y}$ independent?

(c) Are the components of $\mathbf{Y}$ uncorrelated?

4-8. Suppose X_1, ..., X_k are independent binomial random variables with different sample sizes but the same success probability, say X_i has the $\text{Bin}(n_i, p)$ distribution. Show that $Y = X_1 + \ldots + X_k$ has the $\text{Bin}(n_1 + \ldots + n_k, p)$ distribution. Hint: no calculation necessary. This follows from something we already know.

Review Problems from Previous Tests

4-9. Suppose $\mathbf{X}$ is a random vector having mean vector

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and variance matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

(a) Calculate $E(X_1 + 2X_2 + 3X_3)$.

(b) Calculate $\operatorname{var}(X_1 + 2X_2 + 3X_3)$.

4-10. Suppose a basketball player is shooting free throws and the results (success or failure) are considered to be independent and identically distributed Bernoulli random variables with success probability 3/4 (code success as one and failure as zero). The player shoots three free throws. What is the probability that she makes at least one?