Stat 5101 (Geyer) Fall 2008
Homework Assignment 12
Due Wednesday, December 10, 2008

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

12-1. Give the details of the argument that the Pois($\mu$) distribution is approximately normal when $\mu$ is large.

12-2. Suppose $X_1, X_2, \ldots$ are IID with mean $\mu$ and variance $\sigma^2$ and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

What is the approximate normal distribution of $\sin(\overline{X}_n)$ when $n$ is large?

12-3. Suppose $X_1, X_2, \ldots$ are IID Pois($\mu$) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

To what random variable does

$$\sqrt{n}(e^{-\overline{X}_n} - e^{-\mu})$$

converge in distribution?

12-4. Suppose $X_1, X_2, \ldots$ are IID Ber($p$) random variables with $0 < p < 1$ and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

(a) What is the approximate normal distribution of $\overline{X}_n(1 - \overline{X}_n)$ when $n$ is large?

(b) There is something unusual about the case $p = 1/2$. What is that?

12-5. Suppose $X$ is a Pois($\mu$) random variable. For what function $g$ does $g(X)$ have approximate normal distribution for large $\mu$ with variance that is a constant function of the parameter?
12-6. Suppose $X_1, X_2, \ldots$ are IID Exp($\lambda$) random variables and
\[ X_n = \frac{1}{n} \sum_{i=1}^{n} X_i \]
For what function $g$ does $g(X_n)$ have approximate normal distribution for large $n$ with variance that is a constant function of the parameter?

12-7. Show that the $\mathcal{N}(np, n(P - pp^T))$ distribution is degenerate: if $X = (X_1 + \cdots + X_k)$ is a random vector having this distribution, then
\[ X_1 + \cdots + X_k = n \]
almost surely. Here $P - pp^T$ is the variance matrix of the $\text{Multi}(n,p)$ distribution.
Hint: If you want to use matrix notation in this problem, define the vector $u = (1,1,\ldots,1)$ so that
\[ u^T X = X_1 + \cdots + X_k \]

12-8. Suppose $X_1, X_2, \ldots,$ is an IID sequence of random variables, having first four ordinary moments
\[ \alpha_i = E(X_i^i), \quad i = 1, \ldots, 4. \]
Define
\[ Y_n = X_n^2, \quad n = 1, 2, \ldots \]
and
\[ \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \]
\[ \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \]
What is the approximate normal distribution of $\bar{Y}_n - \bar{X}_n^2$ when $n$ is large?
Hint: Slides 87–89, deck 6 and the multivariate delta method.