Stat 8112 (Geyer) Spring 2013
Homework Assignment 9
Optional, no due date.

Most of these were submitted as questions for previous PhD prelims (I don't know how many were actually used). They are the kind of questions I submit for PhD prelims.

9-1. Suppose $\mathbf{X} = (X_1, X_2, X_3)$ has a trinomial distribution with vector of cell probabilities $\mathbf{p} = (p_1, p_2, p_3)$. Consider the submodel with univariate parameter θ defined by

$$\begin{aligned} p_1(\theta) &= \tfrac{1}{4}(2 + \theta) \\ p_2(\theta) &= \tfrac{1}{2}(1 - \theta) \\ p_3(\theta) &= \tfrac{1}{4}\theta \end{aligned}$$

(a) Show that the "usual regularity conditions" for maximum likelihood are satisfied so that the maximum likelihood estimator is consistent and asymptotically normal.

(b) Find the expected Fisher information. (You do not have to find the maximum likelihood estimate, since that would involve solving a cubic equation.)

9-2. Suppose X_1, X_2, $\ldots$, are IID from $\mathcal{N}(\theta, \theta^4)$. Find a consistent and asymptotically efficient estimator of θ with a closed-form expression. (That is, saying "the MLE" won't do unless you can produce a closed-form expression for the MLE.)

9-3. Find the asymptotic normal distribution for the log cross-product ratio in a 2 by 2 contingency table. The table

$$\begin{matrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{matrix}$$

is assumed to be multinomial with sample size n and cell probabilities

$$\begin{matrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{matrix}$$

The log cross-product ratio is

$$Y = \log\left(\frac{X_{11} X_{22}}{X_{12} X_{21}}\right)$$

Derive the asymptotic normal distribution for Y, the asymptotics being as n goes to infinity.

9-4. By an $\text{Exp}(\lambda)$ random variable we mean a nonnegative random variable with density

$$f_\lambda(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

where $\lambda > 0$ is an unknown parameter. Suppose (X_n, Y_n), $i = 1, 2, \ldots$ are independent and identically distributed random vectors where for each n the components X_n and Y_n are independent random variables and X_n is $\text{Exp}(\theta)$ distributed and Y_n is $\text{Exp}(1/\theta)$ distributed.

Suppose we want to do a test of the point null hypothesis $\theta = \theta_0$ versus the alternative $\theta \neq \theta_0$. Three asymptotically equivalent large sample tests are available: the Wilks test (also called likelihood ratio test), the Rao test (also called score test or Lagrange multiplier test), and the Wald test. All three test statistics have an asymptotic chi-square distribution with one degree of freedom. Find all three of these test statistics. (You do not have to find anything else other than the form of these three test statistics for this problem.)

9-5. Suppose X_1, X_2, $\ldots$ are independent and identically distributed random variables having density

$$f_\theta(x) = \frac{\Gamma(2\theta)}{\Gamma(\theta)^2} x^{\theta - 1} (1 - x)^{\theta - 1}, \qquad 0 < x < 1,$$

where θ is an unknown parameter satisfying $\theta > 0$.

(a) Find a method of moments estimator of θ that is a function of X_1, $\ldots$, X_n.

(b) Find the asymptotic distribution of your method of moments estimator.

9-6. Suppose X_1, X_2, $\ldots$ are independent and identically $\text{Normal}(\mu, \sigma^2)$ distributed. Find the asymptotic normal distribution of the estimator S_n of the unknown parameter σ, where

$$S_n^2 = \frac{1}{n - 1} \sum_{i=1}^n (X_i - \overline{X}_n)^2$$

and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

You may take as your starting point the exact sampling distribution

$$\frac{(n - 1) S_n^2}{\sigma^2} \sim \text{ChiSq}(n - 1).$$

9-7. Suppose X_1, X_2, ... are independent and identically distributed normal random variables, having mean θ and variance θ, which requires $\theta \geq 0$. Then

$$\overline{X}_n = \sum_{i=1}^{n} X_i$$

is a consistent and asymptotically normal estimator of θ. Find the asymptotic relative efficiency of $\overline{X}_n$, that is, the ratio of the asymptotic variance of $\overline{X}_n$ to the asymptotic variance of the maximum likelihood estimator (MLE). You do not have to find an expression for the MLE.

9-8. Let (X_i, Y_i), $i = 1, 2, \ldots$ be an independent and identically distributed sequence of random vectors, having second moments. Find the asymptotic normal distribution of

$$\hat{\theta}_n = \sqrt{\overline{X}_n \overline{Y}_n}$$

where

$$\overline{X}_n = (X_1 + \cdots + X_n)/n$$
$$\overline{Y}_n = (Y_1 + \cdots + Y_n)/n$$

9-9. For a random variable X with mean μ and standard deviation σ, the *signal to noise ratio* is $\theta = \mu/\sigma$. For an independent, identically distributed sequence X_1, X_2, ..., the signal to noise ratio can be estimated by $\hat{\theta}_n = \overline{X}_n / S_n$, where $\overline{X}_n$ and S_n are the sample mean and standard deviation.

(a) Find the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

(b) Find a variance-stabilizing transformation for $\hat{\theta}_n$ in the special case where the X_i are i. i. d. normal.

9-10. Suppose X_1, X_2, ..., X_n are independent, identically distributed exponential failure times with mean μ, that is, the density of any of the X_i is

$$f_\mu(x) = \frac{1}{\mu} e^{-x/\mu}, \qquad x > 0.$$

But the X_i are not all observed. The test only runs for a fixed time T, which is known. If the item being tested does not fail by time T, the test does not continue. Thus the observed data are $Y_i = X_i \wedge T$, the minimum of X_i and T.

(a) Write down the log likelihood, and find an explicit formula for the maximum likelihood estimate of the mean time to failure μ.

(b) Find an explicit formula for the expected Fisher information.

(c) Suppose 20 light bulbs are tested for $T = 1000$ hours. 11 bulbs are observed to fail within the 1000 hours. Their failure times are

$$71 \quad 88 \quad 254 \quad 339 \quad 372 \quad 403 \quad 498 \quad 499 \quad 593 \quad 774 \quad 935$$

Find the maximum likelihood estimate of μ and a 95% asymptotic confidence interval for μ.

(d) Show that this satisfies the regularity conditions (1) through (4) of Theorem 18 in Ferguson or regularity conditions (a) through (e) of the preprint about no-n asymptotics.