Stat 8112 (Geyer) Spring 2013
Homework Assignment 8
Due Friday May, 10, 2013

9-1. If K is a kernel, μ is a measure, f is a function, and I is the identity kernel, show

$$
\begin{aligned}
KI &= K \\
IK &= K \\
\mu I &= \mu \\
If &= f
\end{aligned}
$$

9-2. Is the Markov chain with identity transition probability kernel irreducible? Either way show why. Does this depend on what the state space is?

9-3. Consider a finite state space Markov chain whose transition probability matrix has elements p_{ij} that are nonzero if and only if $i = j$ or $i + 1 = j$. Is this Markov chain irreducible? Either way show why.

Warning: In this course *irreducible* means φ-irreducible for some positive measure φ, which is the definition used in the theory of Markov chains on general state spaces. If you use the definition from most stochastic processes courses (which only cover discrete state space Markov chains and which define "irreducible" to mean φ-irreducible where φ is counting measure on the state space), then you won't get the right answers.

9-4. Let f be a joint probability density function on $\mathbb{R}^2$, and define

$$
W = \{\, (x, y) \in \mathbb{R}^2 : f(x, y) > 0 \,\}.
$$

Let $f_{X|Y}$ and $f_{Y|X}$ denote the two conditional probability density functions derived from f. Consider a Markov chain (X_n, Y_n), $n = 1, 2, \ldots$ having state space is W that moves as follows

$$
\begin{aligned}
X_{n+1} &\sim f_{X|Y}(\,\cdot \mid Y_n) \\
Y_{n+1} &\sim f_{Y|X}(\,\cdot \mid X_{n+1})
\end{aligned}
$$

where $\sim$ means "is simulated from the distribution." (For those familiar with the Gibbs sampler, this is a special case.) Suppose that W is a connected open set, and show that the Markov chain is irreducible.

9-5. Let X_1, X_2, ... be Markov chain having the integers as its state space and that moves as follows

$$\Pr(X_{n+1} = X_n + 1) = \Pr(X_{n+1} = X_n - 1) = 1/2.$$

Show that this Markov chain is null recurrent.

9-6. Consider the Markov chain used as an example in Section 3.4 of the handout on Markov chains (that simulates the distribution uniform on an open set W in $\mathbb{R}^d$). Suppose that W is a bounded set, and show that the Markov chain is uniformly ergodic.

9-7. Consider the balanced one-way random effects model. The data Y is an $r \times c$ matrix having components Y_{ij}, which are assumed to be normal with means and variances

$$\begin{aligned} E(Y_{ij} \mid \theta_i) &= \theta_i \\ \operatorname{var}(Y_{ij} \mid \theta_i) &= \lambda^{-1} \end{aligned}$$

The random effects θ_i are assumed to be normal with means and variances

$$\begin{aligned} E(\theta_i) &= 0 \\ \operatorname{var}(\theta_i) &= \nu^{-1} \end{aligned}$$

Let θ denote the vector having components θ_i. The components of Y are conditionally independent given θ, and the components of θ are independent. The parameters λ and ν, which are reciprocals of variances, are called precisions.

Suppose we want to be fully Bayesian about this and hence want to assume priors on the parameters. For reasons of mathematical convenience we assume gamma priors

$$\begin{aligned} \lambda &\sim \text{Gamma}(a_1, b_1) \\ \nu &\sim \text{Gamma}(a_2, b_2) \end{aligned}$$

where a_1, a_2, b_1, and b_2 are known numbers (the hyperparameters). This problem will be less messy if you assume an unnormalized Gamma(a, b) density has the form $h(x) = x^{a-1} \exp(-bx)$.

Show that the components of θ are conditionally independent given λ, ν, and Y and that λ and ν are conditionally independent given θ and Y. This suggests a "block Gibbs" sampler for simulating the posterior distribution of θ, λ, and ν given Y. Each step of the Markov chain does two substeps

- Simulate a new value of θ from the conditional distribution of θ given λ, ν, and Y.

- Simulate new values of λ and ν from their conditional distribution given θ and Y.

The Gibbs sampler is easy to implement because these are "brand name" distributions. Find the conditional distributions for these two substeps.

Show that the resulting Gibbs sampler is geometrically ergodic. It is actually an open research question to show this for all values of the hyperparameters. You are allowed to pick any values of the hyperparameters.