Stat 8112 (Geyer) Spring 2013
Homework Assignment 2
Due Wednesday, February 13, 2013

2-1. Ferguson, Problem 4-1.

2-2. Ferguson, Problem 4-3.

2-3. Ferguson, Problem 4-5.

2-4. Ferguson, Problem 4-6.

2-5. Ferguson, Problem 4-8.

2-6. We know independent and identically distributed (IID) sampling from the Cauchy distribution does not satisfy the strong law of large numbers because first moments of the Cauchy distribution do not exist. Does the Cauchy distribution satisfy the weak law of large numbers? Explain why or why not.

2-7. Prove the statement in Section 7 of the handout on metric spaces that if $d(x, y)$ is a metric, then so is

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Hint: in proving the triangle inequality it may help to consider separately the cases $d(x, z) \geq d(x, y)$ and $d(x, z) \geq d(y, z)$ (that's case 1) and $d(x, z) < d(x, y)$ or $d(x, z) < d(y, z)$ (that's case 2).

2-8. Show that the formula

$$d(f, g) = \sum_{n=1}^{\infty} \frac{2^{-n} \|f - g\|_{B_n}}{1 + \|f - g\|_{B_n}}$$

in Section 12.1 of the handout on metric spaces defines a metric (use the preceding problem as a lemma). Don't forget to prove that the formula defines a real-valued function, that is, don't prove that the infinite series in the formula always converges.

2-9. Find the joint asymptotic distribution of $\widehat{P}_n(B_i)$, $i = 1, \ldots, d$, where $\widehat{P}_n$ is the empirical measure defined in the handout on metric spaces.

Note that the B_i can be any sets. They need not be disjoint or have any other special property. Since d is finite, we are just talking about an IID sequence of random vectors

$$Y_n = \bigl(\widehat{P}_n(B_1), \widehat{P}_n(B_2), \ldots \widehat{P}_n(B_d)\bigr)$$

2-10. Specialize your result in problem 2-9 to the case where $B_1, \ldots, B_d$ are disjoint and their union is the whole metric space. Show that this agrees with the familiar asymptotics for multinomial random vectors. (If it is not familiar, look it up.)

2-11. Suppose $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are IID sequences that are independent of each other. Define

$$\begin{aligned}
\mu_X &= E(X_i) \\
\mu_Y &= E(Y_i) \\
\sigma_X^2 &= \operatorname{var}(X_i) \\
\sigma_Y^2 &= \operatorname{var}(Y_i) \\
\overline{X}_n &= \frac{1}{n} \sum_{i=1}^n X_i \\
\overline{Y}_n &= \frac{1}{n} \sum_{i=1}^n Y_i
\end{aligned}$$

and suppose k_n and m_n are positive-integer-valued sequences that both go to infinity. Carefully show that

$$\frac{(\overline{X}_{k_n} - \overline{Y}_{m_n}) - (\mu_X - \mu_Y)}{\sqrt{\dfrac{\sigma_X^2}{k_n} + \dfrac{\sigma_Y^2}{m_n}}} \xrightarrow{\mathcal{L}} \text{Normal}(0, 1)$$

Hint: first prove this under the additional assumption that

$$\frac{k_n}{k_n + m_n} \to \alpha$$

(you will need different arguments for the three cases $\alpha = 0$, $0 < \alpha < 1$, and $\alpha = 1$). Then use the subsequence principle.

2-12. Ferguson, Problem 5-3.

2-13. Ferguson, Problem 5-4.

2-14. Ferguson, Problem 5-7.