Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

3-1. Show that the family of $\text{Gam}(\alpha, \lambda)$ distributions with $\alpha$ known and $\lambda$ unknown, so the parameter space is

$$\{ \lambda \in \mathbb{R} : \lambda > 0 \}$$

is a scale family.

3-2. Suppose $S_n^2$ is the sample variance calculated from an IID normal random sample of size $n$.

(a) Calculate the bias of $S_n$ as an estimator of the population standard deviation $\sigma$.

(b) Find the constant $a$ such that $aS_n$ has the smallest mean square error as an estimator of $\sigma$.

3-3. Suppose $U$ and $V$ are statistics that are independent random variables and both are unbiased estimators of a parameter $\theta$. Write $\text{var}(U) = \sigma_U^2$ and $\text{var}(V) = \sigma_V^2$, and define another statistic $T = aU + (1 - a)V$ where $a$ is an arbitrary but known constant.

(a) Show that $T$ is an unbiased estimator of $\theta$.

(b) Find the $a$ that gives $T$ the smallest mean square error.

3-4. The slides don't give any examples of estimators that are not consistent. Give an example of an inconsistent estimator of the population mean.

3-5. If $X \sim \text{Bin}(n, p)$, show that $\hat{p}_n = X/n$ is a consistent and asymptotically normal estimator of $p$, and give the asymptotic distribution of $\hat{p}_n$.

3-6. If $X_1, X_2, \ldots$ are IID from a distribution having a variance $\sigma^2$, show that both $V_n$ and $S_n^2$ are consistent estimators of $\sigma^2$. 
3-7. Suppose $X_1, X_2, \ldots$ are IID Geo($p$).

(a) Find a method of moments estimator for $p$.

(b) Find the asymptotic normal distribution of your estimator.

3-8. Suppose $X_1, X_2, \ldots$ are IID Beta($\alpha, 2$).

(a) Find a method of moments estimator for $\alpha$.

(b) Find the asymptotic normal distribution of your estimator.

3-9. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a Beta($\theta, \theta$) model, where $\theta$ is an unknown parameter. Find a method of moments estimator of $\theta$.

Review Problems from Previous Tests

3-10. For the following data

\[
1.5 \quad 2.0 \quad 2.5 \quad 3.0 \quad 4.5
\]

(a) Find the mean of the empirical distribution.

(b) Find the median of the empirical distribution.

(c) Find $P_n(X \leq 3)$ under the empirical distribution.

(d) Find the 0.25 quantile of the empirical distribution.

3-11. Find the asymptotic distribution of the sample median of an IID sample from the Exp($\lambda$) distribution.

3-12. Suppose $X_1, X_2, \ldots$ are IID NegBin($r, p$), where $r$ is known and $p$ is unknown.

(a) Find a method of moments estimator for $p$.

(b) Find the asymptotic normal distribution of your estimator.