1 Calculating an Expectation or a Probability

Probability is a special case of expectation (deck 1, slide 62).
1.1 From a PMF

If $f$ is a PMF having domain $S$ (the sample space) and $g$ is any function, then

$$E\{g(X)\} = \sum_{x \in S} g(x) f(x)$$

(deck 1, slide 56), and for any event $A$ (a subset of $S$)

$$\Pr(A) = \sum_{x \in S} I_A(x) f(x) = \sum_{x \in A} f(x)$$

(deck 1, slide 62).

1.2 From given Expectations using Uncorrelated

If $X$ and $Y$ are uncorrelated random variables, then

$$E(XY) = E(X)E(Y)$$

(deck 2, slide 73).

1.3 From given Expectations using Independent

If $X$ and $Y$ are independent random variables and $g$ and $h$ are any functions, then

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

(deck 2, slide 76).

More generally, if $X_1, X_2, \ldots, X_n$ are independent random variables and $g_1, g_2, \ldots, g_n$ are any functions, then

$$E\left\{ \prod_{i=1}^{n} g_i(X_i) \right\} = \prod_{i=1}^{n} E\{g_i(X_i)\}$$

(deck 2, slide 76).
1.4 From given Expectations using Linearity of Expectation

1.4.1 Expectation of Sum and Average

If $X_1, X_2, \ldots, X_n$ are random variables, then

$$E\left\{ \sum_{i=1}^{n} X_i \right\} = \sum_{i=1}^{n} E(X_i)$$

(deck 2, slide 10). In particular, if $X_1, X_2, \ldots, X_n$ all have the same expectation $\mu$, then

$$E\left\{ \sum_{i=1}^{n} X_i \right\} = n\mu$$

(deck 2, slide 90), and

$$E(\bar{X}_n) = \mu,$$

where

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

(deck 2, slide 90).

1.4.2 Variance of Sum and Average

If $X_1, X_2, \ldots, X_n$ are random variables, then

$$\text{var}\left\{ \sum_{i=1}^{n} X_i \right\} = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{var}(X_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \text{cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} \text{var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \text{cov}(X_i, X_j)$$

(deck 2, slide 71). In particular, if $X_1, X_2, \ldots, X_n$ are uncorrelated, then

$$\text{var}\left\{ \sum_{i=1}^{n} X_i \right\} = \sum_{i=1}^{n} \text{var}(X_i)$$
(deck 2, slide 75). More particular, if \( X_1, X_2, \ldots, X_n \) are uncorrelated and all have the same variance \( \sigma^2 \), then

\[
\text{var} \left\{ \sum_{i=1}^{n} X_i \right\} = n \sigma^2
\]

(deck 2, slide 90) and

\[
\text{var}(\bar{X}_n) = \frac{\sigma^2}{n}
\]

(deck 2, slide 90), where \( \bar{X}_n \) is given by (1).

### 1.4.3 Expectation and Variance of Linear Transformation

If \( X \) is a random variable and \( a \) and \( b \) are constants, then

\[
E(a + bX) = a + bE(X)
\]

\[
\text{var}(a + bX) = b^2 \text{var}(X)
\]

(deck 2, slide 8 and slide 37).

### 1.4.4 Covariance of Linear Transformations

If \( X \) and \( Y \) are random variables and \( a, b, c, \) and \( d \) are constants, then

\[
\text{cov}(a + bX, c + dY) = bd \text{cov}(X, Y)
\]

(homework problem 3-7).

### 1.4.5 Expectation and Variance of Vector Linear Transformation

If \( \mathbf{X} \) is a random vector, \( \mathbf{a} \) is a constant vector, and \( \mathbf{B} \) is a constant matrix such that \( \mathbf{a} + \mathbf{B}\mathbf{X} \) makes sense (the dimension of \( \mathbf{a} \) and the row dimension of \( \mathbf{B} \) are the same, and the dimension of \( \mathbf{X} \) and the column dimension of \( \mathbf{B} \) are the same), then

\[
E(\mathbf{a} + \mathbf{B}\mathbf{X}) = \mathbf{a} + \mathbf{B}E(\mathbf{X})
\]

\[
\text{var}(\mathbf{a} + \mathbf{B}\mathbf{X}) = \mathbf{B} \text{var}(\mathbf{X})\mathbf{B}^T
\]

(deck 2, slide 64).
1.4.6 “Short Cut” Formulas

If $X$ is a random variable, then

$$\text{var}(X) = E(X^2) - E(X)^2$$

(deck 2, slide 21).

If $X$ and $Y$ are random variables, then

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y)$$

(homework problem 3-7).

2 Change of Variable Formulas

2.1 Discrete Distributions

2.1.1 One-to-one Transformations

If $f_X$ is the PMF of the random variable $X$, if $Y = g(X)$, and $g$ is an invertible function with inverse function $h$ (that is, $X = h(Y)$), then

$$f_Y(y) = f_X[h(y)]$$

and the domain of $f_Y$ is the range of the function $g$ (the set of possible $Y$ values) (deck 2, slide 86).

2.1.2 Many-to-one Transformations

If $f_X$ is the PMF of the random variable $X$ having sample space $S$ and $Y = g(X)$, then

$$f_Y(y) = \sum_{x \in S} f_X(x)$$

and the domain of $f_Y$ is the codomain of the function $g$ (deck 2, slide 81).

3 PMF and Independence

If $X_1, \ldots, X_n$ are independent random variables having PMF $f_1, \ldots, f_n$, then

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i)$$
In particular, if $X_1, \ldots, X_n$ are independent and identically distributed random variables having PMF $h$, then

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} h(x_i).$$