Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

3-1. Suppose that $f$ is a PMF on a sample space $S$, suppose $X$ and $Y$ are random variables in this probability model. Prove the following statements.

(a) $E(X + Y) = E(X) + E(Y)$.
(b) If $X(s) \geq 0$ for all $s \in S$, then $E(X) \geq 0$.
(c) If $Y(s) = a$ for all $s \in S$, then $E(XY) = aE(X)$.
(d) If $Y(s) = 1$ for all $s \in S$, then $E(Y) = 1$.

Do not use the axioms (these are the axioms). The problem is to prove that these statements follow from our earlier definition of PMF and expectation.

3-2. Suppose $X$ has the uniform distribution on the set \{1, 2, 3, 4\}, and suppose $Y = X^2$.

(a) Calculate $E(X)$.
(b) Calculate $E(Y)$.
(c) Calculate $E(Y/X)$.
(d) Calculate $E(Y)/E(X)$.
(e) Compare your answers in (c) and (d). Are they the same? Should they be the same?


3-4. Suppose $X$ is a random variable having PMF given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1/9</td>
<td>2/9</td>
<td>3/9</td>
<td>2/9</td>
<td>1/9</td>
</tr>
</tbody>
</table>

(a) Calculate $E(X)$.
(b) Calculate $\text{var}(X)$.
3-5. Suppose $X$ is a Ber($p$) random variable and $Y = 2X - 1$.
(a) Calculate $E(Y)$.
(b) Calculate $\text{var}(Y)$.
(c) Calculate $E(Y^2)$.
(d) Calculate $\text{var}(Y^2)$.

3-6. Suppose $X$ has the discrete uniform distribution on the set
$$\{ x \in \mathbb{Z} : l \leq x \leq u \}$$
where $l$ and $u$ are integers with $l < u$.
(a) Calculate $E(X)$.
(b) Calculate $\text{var}(X)$.

3-7. Suppose $X$ and $Y$ are random variables in the same probability model, and suppose $a$, $b$, $c$, and $d$ are constants.
(a) Prove that
$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$
(b) Prove that
$$\text{cov}(a + bX, c + dY) = bd\text{cov}(X,Y)$$

3-8. Suppose
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$
(a) Calculate $\mathbf{A}\mathbf{B}$.
(b) Calculate $\mathbf{B}\mathbf{A}$.
(c) Compare your answers in (a) and (b). Are they the same? Should they be the same?

3-9. Suppose $X$ is a random variable with mean $\mu$ and variance $\sigma^2$. Calculate the mean vector and variance matrix of the random vector $\mathbf{Y} = (X, 2 + 3X)$.
3-10. Suppose \( X \) is a random variable with mean vector
\[
\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}
\]
and variance matrix
\[
M = \begin{pmatrix} v_1 & c \\ c & v_2 \end{pmatrix}
\]
Calculate the mean vector and variance matrix of the random vector \( Y = a + BX \) where
\[
a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}
\]

Review Problems from Previous Tests

3-11. Suppose \( X \) is a random variable having probability mass function (PMF) given by
\[
\begin{array}{c|ccccc}
  x & -2 & -1 & 0 & 1 & 2 \\
  f(x) & 1/3 & 1/6 & 0 & 1/6 & 1/3 \\
\end{array}
\]
(a) Calculate \( E(X) \).
(b) Calculate \( \text{var}(X) \).

3-12. Suppose \( X \) is a random variable having PMF given by
\[
\begin{array}{c|ccccc}
  x & -2 & -1 & 0 & 1 & 2 \\
  f(x) & 1/9 & 2/9 & 1/3 & 2/9 & 1/9 \\
\end{array}
\]
(a) Find the PMF of the random variable \( Y = X^2 \).
(b) Calculate \( \Pr(Y > 0) \).

3-13. Suppose the random vector \((X, Y)\) has PMF given by
\[
f(x, y) = \frac{x^2 y}{90}, \quad x = -2, -1, 0, 1, 2, \ y = 2, 3, 4.
\]
Are \( X \) and \( Y \) independent random variables? Explain why or why not, as the case may be.