Solutions to Final Exam

1. (a) The expected sum is $E[\sum X_i] = n\mu = 64 \times 84 = 5376$, and the standard deviation of the sum is $\mathrm{SD}(\sum X_i) = \sqrt{\mathrm{Var}(\sum X_i)} = \sqrt{n\sigma^2} = \sqrt{n}\sigma = 8 \times 8 = 64$.

   (b) The standard deviation of the sample average is $\sigma/\sqrt{n} = 8/8 = 1$. By the central limit theorem the sample average is approximately normally distributed, so

$$\begin{aligned}
\Pr(|\overline{X} - \mu| \leq 1) &\approx \Pr(|Z| \leq 1) = \Pr(Z \leq 1) - \Pr(Z \leq -1) \\
&= \Pr(Z \leq 1) - (1 - \Pr(X \leq 1)) = 2\Pr(Z \leq 1) - 1 \\
&= 2 \times 0.8413 - 1 = 0.6826.
\end{aligned}$$

2. (a) The mean of X is

$$E[X] = \int_0^1 x^2 + \frac{1}{2} x dx = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

   The expected value of X^2 is

$$E[X^2] = \int_0^1 x^3 + \frac{1}{2} x^2 dx = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

   The variance of X is

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{60}{144} - \frac{49}{144} = \frac{11}{144}$$

   (b) For $0 < x < 1$ the distribution function is

$$F(x) = \int_0^x u + \frac{1}{2} du = \frac{x^2}{2} + \frac{x}{2} = \frac{1}{2} x(x+1)$$

   The complete distribution function is therefore

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{2} x(x+1) & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geq 1. \end{cases}$$

3. (a) The means and variances are

$$\begin{aligned}
E[Y_1] &= \mu_1 + \mu_3 & \mathrm{Var}(Y_1) &= \sigma_1^2 + \sigma_3^2 \\
E[Y_2] &= \mu_2 + \mu_3 & \mathrm{Var}(Y_2) &= \sigma_2^2 + \sigma_3^2
\end{aligned}$$

   (b) The covariance is

$$\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= \mathrm{Cov}(X_1 + X_3, Y_2) = \mathrm{Cov}(X_1, Y_2) + \mathrm{Cov}(X_3, Y_2) \\
&= \mathrm{Cov}(X_1, X_2 + X_3) + \mathrm{Cov}(X_3, X_2 + X_3) \\
&= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_3, X_2) + \mathrm{Cov}(X_3, X_3) \\
&= 0 + 0 + 0 + \mathrm{Var}(X_3) = \sigma_3^2
\end{aligned}$$

(c) If $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma^2$ then $\text{Var}(Y_1) = \text{Var}(Y_2) = 2\sigma^2$ and $\text{Cov}(Y_1, Y_2) = \sigma^2$. So the correlation is

$$\rho = \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{\text{Var}(Y_1)\text{Var}(Y_2)}} = \frac{\sigma^2}{2\sigma^2} = \frac{1}{2}$$

4. The conditional probability function of Y given $X = n$ is

$$\begin{aligned} f_{Y|X}(y|n) = \Pr(Y = y|X = n) &= \frac{\Pr(Y = y, X = n)}{\Pr(X = 2)} \\ &= \frac{\Pr(Y = y, X - Y = n - y)}{\Pr(X = n)} \end{aligned}$$

For a Poisson process the number of events Y in $[0, s]$ and the number of events $X - Y$ in $(s, t]$ are independent Poisson variables with means rs and $r(t - s)$ and the number of events X in $[0, t]$ is Poisson with mean rt, respectively. So for $y = 0, \ldots, n$

$$\begin{aligned} f_{Y|X}(y|n) &= \frac{\Pr(Y = y)\Pr(X - Y = n - y)}{\Pr(X = n)} \\ &= \frac{\frac{(rs)^y}{y!}e^{-rs}\frac{(r(t-s))^{(n-y)}}{(n-y)!}e^{-r(t-s)}}{\frac{(rt)^n}{n!}e^{-rt}} \\ &= \frac{n!}{y!(n-y)!}\frac{s^y(t-s)^{n-y}}{t^n} = \binom{n}{y}\left(\frac{s}{t}\right)^y\left(1 - \frac{s}{t}\right)^{n-y} \end{aligned}$$

This is a binomial distribution with n trials and success probability s/t.

An alternative approach: Given that there are n events in $[0, t]$, the times at which these events occur are distributed as n independent uniform random variables on $[0, t]$. The probability that a uniform random variable on $[0, t]$ is less than s is s/t, so the number of these times that are less than s is binomial with n trials and success probability s/t.

5. Suppose $N = n$ customers arrive at the store. Since customers act independently, the conditional distribution of X given $N = n$ is binomial(n,p). Thus

$$\begin{aligned} f(x, y) &= \Pr(X = x, Y = y) = \Pr(X = x, X + Y = x + y) \\ &= \Pr(X = x, N = x + y) = \Pr(X = x|N = x + y)\Pr(N = x + y) \\ &= \binom{x+y}{x}p^x(1-p)^{x+y-x}\frac{\lambda^{x+y}}{(x+y)!}e^{-\lambda} = \frac{(x+y)!}{x!y!}p^x(1-p)^y\frac{\lambda^{x+y}}{(x+y)!}e^{-\lambda} \\ &= \frac{(p\lambda)^x}{x!}e^{-p\lambda}\frac{((1-p)\lambda)^y}{y!}e^{-(1-p)\lambda} \end{aligned}$$

Thus X and Y are independent Poisson random variables with means $p\lambda$ and $(1-p)\lambda$, respectively.

6. Let $V = Y$. Then $X = UV$ and $Y = V$. The Jacobian determinant is

$$J(u, v) = \det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} = v$$

The range of the transformation is $\mathcal{T} = (0,\infty) \times (0,\infty)$. For $(u,v) \in \mathcal{T}$ the joint density of U, V is

$$\begin{aligned} f_{UV}(u,v) = v f_X(uv) f_Y(v) &= v \frac{1}{\Gamma(\alpha_1)} (uv)^{\alpha_1 - 1} e^{-uv} \frac{1}{\Gamma(\alpha_2)} v^{\alpha_2 - 1} e^{-v} \\ &= \frac{u^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1 + \alpha_2 - 1} e^{-v(1+u)} \end{aligned}$$

The marginal density of U is therefore

$$\begin{aligned} f_U(u) &= \frac{u^{\alpha_1 - 1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^\infty v^{\alpha_1 + \alpha_2 - 1} e^{-v(1+u)} \\ &= \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{u^{\alpha_1 - 1}}{(1+u)^{\alpha_1 + \alpha_2}} \end{aligned}$$

for $u > 0$ and $f_U(u) = 0$ for $u \leq 0$.

Summary of Scores

Mean	49.2
SD	7.2
Min	33
Lower Quartile	44
Median	49
Upper Quartile	55.5
Max	60