# Notes for Statgen seminar on Scurrah, Palmer, and Burton (2000)

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March 1, 2005

We start with "Model 1" from Scurrah, Palmer, and Burton (2000)

$$\lambda_j(t) = \lambda_0(t) \exp(\beta^T Z_j). \tag{1}$$

A helpful e-mail from Dr. Burton pointed me to a derivation in Aitkin, Anderson, Francis, and Hinde (1989, pp. 287–289) giving the connection between this "Model 1" and "Model 2" of the same paper.

The likelihood for the $j$-th individual is

$$L_j(\beta, \lambda_0) = \begin{cases} f_j(t_j), & \text{if individual } j \text{ is observed to fail at time } t_j \\ 1 - F_j(t_j), & \text{if individual } j \text{ is censored at time } t_j \end{cases}$$

where $f_j$ and $F_j$ are the PDF and CDF of the density of the failure time for the $j$-th individual.

In general (ignoring for the moment this particular problem), the relation between probability functions ($f$ and $F$) and hazard functions ($\lambda$ and $\Lambda$) is

$$\lambda(t) = \frac{f(t)}{1 - F(t)} \tag{2a}$$

and

$$\Lambda(t) = -\log\bigl[1 - F(t)\bigr] \tag{2b}$$

where $\Lambda$ is the cumulative hazard function

$$\Lambda(t) = \int_{-\infty}^{t} \lambda(u)\, du.$$

That (2a) and (2b) agree is easily seen by differentiating (2b). Solving these for $f$ and $F$, we get

$$1 - F(t) = \exp\bigl(-\Lambda(t)\bigr) \tag{3a}$$

and

$$\begin{aligned} f(t) &= \lambda(t)\bigl[1 - F(t)\bigr] \\ &= \lambda(t) \exp\bigl(-\Lambda(t)\bigr). \end{aligned} \tag{3b}$$

Thus, now returning to Scurrah, et al. (2000), we can write the likelihood for the $j$-th individual as

$$L_j(\beta, \lambda_0) = \lambda_j(t_j)^{w_j} \exp\bigl[-\Lambda_j(t_j)\bigr] \tag{4}$$

where

$$w_j = \begin{cases} 1, & \text{if individual } j \text{ is observed to fail} \\ 0, & \text{if individual } j \text{ is censored} \end{cases}$$

The cumulative hazard in (4) satisfies

$$\Lambda_j(t_j) = \Lambda_0(t_j) \exp(\beta^T Z_j) \tag{5}$$

if we assume the covariates $Z_j$ are time-independent (an important assumption, everything is much more complicated without it).

Now (4) is a very difficult equation to deal with ($\lambda_0$ is an infinite-dimensional parameter). Thus we make $\lambda_0$ finite-dimensional by assuming it is piecewise constant

$$\lambda_0(t) = \lambda_{0k} \qquad \tau_{k-1} < t \le \tau_k.$$

This model is called "piecewise exponential" by Aitkin, et al. (1989) because the survival time density $f$ is piecewise exponential (constant hazard gives exponential density, piecewise constant, piecewise exponential) and is attributed by them to Breslow (1972, 1974). It has cumulative baseline hazard

$$\Lambda_0(t) = \sum_{k=1}^{N} \lambda_{0k} \bigl[(\tau_k - \tau_{k-1}) I(\tau_k < t_j) + (t - \tau_{k-1}) I(\tau_{k-1} < t_j \le \tau_k)\bigr]$$

Defining, following Scurrah, et al. (2000),

$$t_{jk} = \begin{cases} \tau_k - \tau_{k-1}, & \tau_k < t_j \\ t_j - \tau_{k-1}, & \tau_{k-1} < t_j \le \tau_k \\ 0, & t_j \le \tau_{k-1} \end{cases}$$

we get

$$\Lambda_0(t_j) = \sum_{k=1}^{N} \lambda_{0k} t_{jk}. \tag{6}$$

Also defining, again following Scurrah, et al. (2000),

$$d_{jk} = \begin{cases} 1, & \text{the } j\text{-th individual fails in the } k\text{-th interval } (\tau_{k-1}, \tau_k] \\ 0, & \text{otherwise} \end{cases}$$

we get

$$\lambda_j(t_j)^{w_j} = \prod_{k=1}^{N} \bigl(\lambda_{0k} \exp(\beta^T Z_j)\bigr)^{d_{jk}} \tag{7a}$$

(at most one term is not equal to one) and

$$\exp\left[-\Lambda_j(t_j)\right] = \prod_{k=1}^{N} \exp\left[-\exp(\beta^T Z_j)\lambda_{0k} t_{jk}\right] \tag{7b}$$

Now if we define

$$\log(\mu_{jk}) = \log(t_{jk}) + \log(\lambda_{0k}) + \beta_T Z_j$$

as in "Model 2" of Scurrah, et al. (2000) so

$$\mu_{jk} = \lambda_{0k} t_{jk} \exp(\beta^T Z_j)$$

and plug this into (7a) and (7b) and plug them into (4) we get

$$L_j(\beta, \lambda_0) = \prod_{k=1}^{N} \left(\frac{\mu_{jk}}{t_{jk}}\right)^{d_{jk}} \exp(-\mu_{ij}) \tag{8}$$

or, dropping multiplicative factors ($t_{jk}$) that do not contain parameters,

$$L_j(\beta, \lambda_0) = \prod_{k=1}^{N} \mu_{jk}^{d_{jk}} \exp(-\mu_{ij}) \tag{9}$$

is also the likelihood for the $j$-th individual. Individuals being independent, the full likelihood is just the product of (9) over all $j$.

Note that we now have the following answers to the questions that arose during our discussion

- each term of (9) has the form of a Poisson likelihood with mean parameter $\mu_{jk}$. Thus if we tell a program like BUGS that it is a Poisson likelihood, no mistake is made. This is purely a "computer-friendly" description of the problem in terms the computer understands and of no statistical interest.
- the product in (9) does not arise from independence. One can argue that it does arise from a sort of conditional independence inherent in the fact that you only die once. But really we made no explicit use of this. It arose directly from the arithmetic rule governing the exponential of a sum.

## References

Aitkin, M., Anderson, D., Francis, B. and Hinde, J. (1989). *Statistical Modelling in Glim.* Oxford University Press.

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