Likelihood and (a little on) Sufficiency

1 Likelihood

A way to help us determine whether one probability model or another is better.

1.1 Two possible dice, rolling one of them once

Consider two dice; the first has 4 0's and 2 1's, the second has three of each. Then the probability tables for these dice are as follows:

	roll a 0	roll a 1
Die 1	2/3	1/3
Die 2	1/2	1/2

Say I choose to roll die 1. What's the probability of a 0? a 1?

But what if I don't know what die I rolled, but I do know what the number is that was rolled? This is what usually happens in an experiment; we know what our results are, but we don't know which probability model to use to model the population.

Say I roll a 0. If we knew I had rolled die 1, what's the probability of that 0? What if I'd rolled die 2? Which die do you think it was more likely that I rolled?

Let's rewrite this example in terms of a Bernoulli model; we can do this because there are two outcomes (success is rolling a 1, failure is rolling a 0). Then there are two possible values for p, the probability of a success: 1/3 and 1/2. We'll let X refer to the unknown random value that we roll, and x refer to the possible values it can be, either 0 or 1.

Then the probability function for a Bernoulli is

$$P(X = x|p) = p^x(1-p)^{1-x}.$$

If you need to, check that this formula gives you the same table as above when using $p = 1/3$ and $p = 1/2$. This one formula gives us two different probability models, one when $p = 1/3$ and one when $p = 1/2$.

Let's repeat the calculations from above using this model. If $p = 1/3$, what's the probability of a 0? a 1?

Say the outcome is a 0. If we knew $p = 1/3$, what's the probability of that 0? What if $p = 1/2$? Which value of p do you think is more likely?

1.2 Two possible dice, rolling one of them twice.

So far, I've only rolled the die once. What if I roll the same die twice (and record both rolls)?

Say I choose to roll die 1. What's the probability of rolling a 0 and then a 1? Remember that the rolls are independent, so you can multiply the probabilities.

Say I choose to roll die 2. What's the probability of rolling a 0 and then a 1?

Now say I rolled a 0 and then a 1, but I didn't know which die I used. Which die do you think is more likely?

Now let's rewrite this example in terms of our Bernoulli model. Now we have two unknown random values, let's call them X_1 and X_2 and let x_1 and x_2 refer to the possible values they can be, either 0 or 1.

What's the probability function for X_1? for X_2?

What's the joint probability function for X_1 and X_2? Remember they are independent, so you can just multiply.

Can you verify the probabilities you got earlier using these functions?

1.3 Infinite possibilities?

Now, what if we want to consider all possible values of p, not just 1/3 and 1/2? That is, let's consider all possible p values between 0 and 1. (Thinking of them as dice isn't helpful anymore. Perhaps funny shaped coins instead?) There are an infinite number of possibilities, so we can no longer compute the probabilities for each value of p, so we need to look at the function in general. In this example, we'll draw a graph to do this. We'll learn other techniques later.

This is the joint probability function you should have gotten in the last part.

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2|p) &= p^{x_1}(1-p)^{1-x_1}p^{x_2}(1-p)^{1-x_2} \\ &= p^{x_1+x_2}(1-p)^{2-(x_1+x_2)} \end{aligned}$$

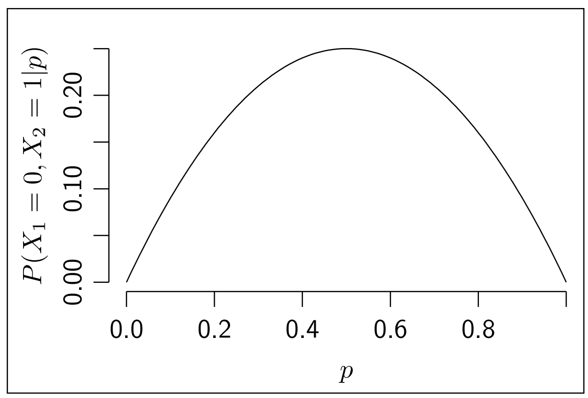
This is now a function only of p, because x_1 and x_2 are our sample data, which is known. We call this new function the *likelihood function*, and we'll call it $L(p)$, or $L(p|\mathbf{X})$ if we want to write it in general, so we can substitute in our actual data later.

Say I rolled a 0 and then a 1, as in the last example. Rewrite this probability function with these known numbers. It should only depend on p! A plot of this function is on the right.

$$P(X_1 = 0, X_2 = 1|p) =$$

Suppose we only considered $p = 1/3$ and $p = 1/2$. Using the graph, which is more likely? Is this the same answer you had in the last part?

Over all the possible values of p, which do you think is the most likely? Are other values of p possible? Would you be surprised if p turned out to really be 0.4? How about 0.1? Or 0?



1.4 Many rolls

Can you write the likelihood function if I rolled the die n times, instead of just twice?

2 Sufficiency

Often, the likelihood will not depend on every data point in the sample. For example, in the above example, the likelihood function only had terms of $x_1 + x_2$ in it, so in order to write the likelihood function we only needed the sum of the rolls, not the actual rolls themselves. The sum is *sufficient* for writing down the likelihood.