1. a) We are counting events (goals scored) in a fixed time period, so it is Poisson. The rate λ is 3 (goals per game), and the time period of interest is $t = 1/2$ (game), so the mean parameter is $\lambda t = 3/2$: $X \sim \text{Poi}(3/2)$

   b) The gender of a child is a Bernoulli random variable, with probability 1/2. Since the couple wants girls, we'll think of getting a girl as a "success." In this case, we are sample from this Bernoulli variable until we get a fixed number of successes, and measuring the number of trials, which is a Negative Binomial; the fixed number of successes is $r = 2$, and $p = 1/2$: $X \sim \text{NegBin}(2, 1/2)$.

   c) We are waiting for a single event to take place, so it is either a Geometric or Exponential, depending if we think of days as continuous or discrete. The rate given is two per week, but since we want it in terms of days, we convert this to 2/7 per day, so either $X \sim \text{Geo}(2/7)$ or $X \sim \text{Exp}(2/7)$.

   d) A bulb is either defective or not, so this is a Bernoulli population; we are sampling a fixed number without replacement, so it is Hypergeometric, with population size $N = 50$, true number of defectives $M = 3$, and sample size $n = 5$.

   e) A nail is either defective or not, so this is again a Bernoulli population. The carpenter is sampling without replacement, as he can't end up with 10 nails to use if he keeps putting them back! Since he samples until he gets a fixed number of good ones and counts the total number of samples, this is a negative hypergeometric, with population size $N = 100$, number of good nails $M = 95$, and waiting for $r = 10$ successes.

2. For $X \sim \text{Bin}(6, 2/3)$, $EX = np = 6(2/3) = 4$, and

$$\text{sd}(X) = \sqrt{\text{Var}(X)} = \sqrt{npq} = \sqrt{6(2/3)(1/3)} = \sqrt{4/3}.$$

To find $P(X \leq 5)$, we use the tables in the back, but since the tables don't have values for $p > 1/2$, we need to think of failures as successes and successes as failures. Since the probability of a failure is 1/3, this is in the table. 5 or less successes is the same as 1 or more failure, so we use the cumulative table (Table Ib), with n=6, k=1, and p=1/3. The answer is .9122.

3. For $X \sim \text{Ber}(p)$, $P(X = 1) = p$ and $P(X = 0) = 1 - p$, the pgf is

$$\eta_X(t) = E(t^X) = t^0(1-p) + t^1(p) = 1 - p + pt.$$

To find the variance, we need to evaluate the first two derivatives (with respect to t)

at $t = 1$; the derivatives are $\eta'_X(t) = p$ and $\eta''_X(t) = 0$, so $\eta'_X(1) = p$ and $\eta''_X(1) = 0$. We know $\eta'_X(1) = EX$ and $\eta''_X(1) = E(X(X-1)) = EX^2 - EX$, so the variance is

$$\begin{aligned}\operatorname{Var}(X) &= EX^2 - (EX)^2 = EX^2 - EX + EX - (EX)^2 \\ &= \eta''_X(1) + \eta'_X(1) - (\eta'_X(1))^2 = 0 + p - p^2 \\ &= p(1-p).\end{aligned}$$

4. Let X be the number of customers entering the store in the first half hour. We are counting events of rate $\lambda = 3$ over a time period $t = 1/2$, so $X \sim \text{Poi}(3/2)$. Then we want to find $P(X \geq 1)$, which is the same is $1 - P(X = 0)$, which is $1 - (3/2)^0 e^{-3/2}/0! = 1 - e^{-3/2} \approx 11/14$.

   Or let X be the amount of time from the store opening until the first customer comes in, so we are measuring time until the first event, so $X \sim \text{Exp}(3)$. Then we want to find $P(X < 1/2) = 1 - F(1/2) = 1 - e^{-3/2} \approx 11/14$.

   Whether at least one customer enters the store or not is a random variable with two states (yes or no), so this is a Bernoulli with probability approximately 11/14. We are repeating this seven times (over the seven days and counting the number of successes, so $Y \sim \text{Ber}(7, 11/14)$, which has an expected of $7(11/14) = 51/2$.

5. a) $P(X < 2) = P(X \leq 2) - P(X = 2) = F(2) - P(X = 2) = 3/4 - 0 = 3/4$; $P(X = 2) = 0$ because there is no jump at $X = 2$.

   b) $P(X = 1)$ is equal to how much the function "jumps" at 1; it jumps from 1/4 to 1/2, so $P(X = 1) = 1/4$.

   c) $P(1 < X < 3) = P(X < 3) - P(X \leq 1) = P(X \leq 3) - P(X = 3) - P(X \leq 1)$
   $= F(3) - P(X = 3) - F(1) = 1 - 1/4 - 1/2 = 1/4$.
   You can also think of this probability as the amount F increases from just after 1 to just before 3; that's from 1/2 to 3/4, so it increases 1/4.

6. a) $1 = \displaystyle\int_0^1 cx\,dx = \frac{cx^2}{2}\Big|_0^1 = c/2$, so $c = 2$.

   b) $EX = \displaystyle\int_0^1 x(2x)\,dx = \frac{2x^3}{3}\Big|_0^1 = 2/3$.

   c) $\operatorname{Var}(X) = EX^2 - (EX)^2$. $EX^2 = \displaystyle\int_0^1 x^2(2x)\,dx = \frac{2x^4}{4}\Big|_0^1 = 1/2$, so
   $\operatorname{Var}(X) = 1/2 - (2/3)^2 = 1/18$.

d) $F(x) = P(X \le x)$. Because all the probability is between 0 and 1, for $x < 0$, $P(X \le x) = 0$, and for $x > 1$, $P(X > x) = 1$.
For $0 \le x \le 1$, $P(X \le x) = \int_0^x 2x\,dx = \frac{2x^2}{2}\Big|_0^x = x^2$. So

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ x^2 & \text{for} \quad 0 \le x \le 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

7. a) $f(y) = \int_y^\infty 2e^{-x-y}\,dx = -2e^{-x-y}\Big|_{x=y}^{y=\infty} = 2e^{-2y}$, for $y > 0$.

b) This region is entirely contained within the support, and it is rectangular, so the bounds are simply X from 1 to ∞ and Y from 0 to 1.

$$\begin{aligned} P(X > 1, Y < 1) &= \int_1^\infty \int_0^1 2e^{-x-y}\,dy\,dx \\ &= 2\int_1^\infty \int_0^1 e^{-x}e^{-y}\,dy\,dx \\ &= 2\left(\int_1^\infty e^{-x}\,dx\right)\left(\int_0^1 e^{-y}\,dy\right) \\ &= 2\left(-e^{-x}\Big|_1^\infty\right)\left(-e^{-y}\Big|_0^1\right) \\ &= 2e^{-1}(1 - e^{-1}) \end{aligned}$$

c) Since the support is not rectangular, but triangular, they must not be independent. For example, at the point $(1, 3)$, the joint density is 0 because $y > x$. But the marginal for y and x are positive for any positive values, so the joint cannot equal the product of the marginals at this point.